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ELEMENTS OF ALGEBRA.

THE

ELEMENTS OF ALGEBRA

DESIGNED FOR

THE USE OF SCHOOLS,

BY THE

REV. J. W. COLENSO, M.A.

FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE,
AND LATE MATHEMATICAL MASTER AT HARROW SCHOOL.

Third Edition,
REVISED, AND IMPROVED.

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PREFACE.

A Third Edition of this little work being called for, the Author wishes to take this opportunity of stating briefly the reasons which led to its first publication, as well as of noticing the principal changes which have been made in the contents and arrangement of the present volume, and, especially, of explaining its diminished size in comparison with that of its predecessor.

It required no very long experience, as Mathematical Master of a Public School, to convince him of the great difficulty of teaching to pupils of this class even the elements of Algebra, as required in the University, with the books generally used. Some were wanting in clearness and perspicuity, and ill adapted to the style of modern analysis; others, more simple and lucid in their explanations, were, however, large and cumbrous, (a serious evil among schoolboys), and besides contained much that was altogether beyond their grasp or comprehension; all were very defective, as probably most mathematical teachers will have felt, in not supplying an adequate number of examples, sufficiently varied and amusing to encourage the efforts of the young student, and sufficiently easy to allow of his managing them with little help from his teacher, and thus acquiring a practical acquaintance with the simpler operations and processes of the science. Without an abundance of these, little can generally be done with pupils at this age; and it is in this way only they are for the most part found *willing*, and sometimes, perhaps, *able*, to imbibe analytical knowledge.

With the view, therefore, of remedying this defect, the Author prepared the first edition of this work, to the end of *Simple Equations;* which was subsequently completed to the full extent of what he deems essential to be known as Elementary Algebra, and sufficient for all the purposes of preparatory education, and even for an ordinary degree at the University.

He has had an opportunity of testing for some time its character as a school book; and the result of his own experience has suggested the alterations which have been made in this edition. These, he believes, will be considered, by those who may have employed the former edition, to be so decidedly improvements for the end in question, as, together with the reduction in price which the present permits, to excuse, he trusts, the difficulty which may at first arise from the changed appearance of the volume itself.

The diminution in size has been attained, not by omission of any thing essential in the former edition, but by a very careful revision, removal of unnecessary remarks and useless examples, and close attention to condensation in the printing; by which means, although additions have been made on the *Least Common Multiple*, *Indeterminate Equations*, and the *Binomial Theorem*, the bulk of the book has been reduced nearly one half. The arrangement of some of the chapters has been modified, that on Surds being placed immediately after Equations; and Euler's proof of the Binomial for Fractional and Negative Indices has been given, as being, after all, the most simple and concise, and likely to be intelligible to any boy of good understanding.

Should any future edition be rendered necessary, the Author hopes that the care bestowed on the revision of this will secure the book from any material change.

ST. JOHN'S COLLEGE,
Sept. 1, 1842.

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The Student is requested, *before using this book,* to correct the following

ERRATA.

PAGE	
52.	*Place* 'Find the cube roots' *after* Ex. 224.
—	In Ex. 226, *write* 7077888, 56.623104.
94.	In *lines* 12, 20, and in Note, *write* (56) for (55).
115.	In *line* 7, *write* (56) *for* (55).
	In *Ans.* 448, *write* $\frac{2}{3a^{\frac{3}{2}}b^{-2}c^{-2}}$.

BOOK-WORK ARTICLES.

18, 31, 32, 36, 37, 50, 51, 56, 79, 83—85, 92, 95—100, 112—114, 121—128, 130, 132, 133, 136—138, 141—143, 147, 149—157, 159—163, 167—169, 171—175, 177—182, 184—195.

ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. ALGEBRA is the science which reasons about quantities by means of letters of the Alphabet, and certain signs and symbols, which are employed to represent both the quantities themselves, and the manner in which they are connected with others.

Thus we might put a to represent 7, and then *twice* a would represent 14; or we might put a to represent 3, and then *twice* a would represent 6, *three times* a, 9, &c.

2. The sign = (*equal*) denotes that the quantities between which it stands are equal to one another.

Thus, if $a = 17$, then *twice* $a = 34$.

3. The sign ∴ stands for *then* or *therefore*, and ∵ for *since* or *because*.

4. The sign + (*plus*) denotes that the quantity before which it stands is *added*, and the sign − (*minus*) that the quantity before which it stands is *subtracted*.

Thus $5 + 3 = 8$, $5 - 3 = 2$; and if $a = 3$ and $b = 4$,

we have $a + b = 3 + 4 = 7$,

$$a + b + 2 = 3 + 4 + 2 = 9,$$

$$5 - a = 5 - 3 = 2,$$

$$10 - a - b = 10 - 3 - 4 = 7 - 4 = 3.$$

5. All quantities before which + stands are called *positive*, and all before which – stands are called *negative* quantities.

If neither + nor – stand before a quantity, + is always understood, and the quantity is positive; thus a means $+ a$.

6. The sign × (*into*) denotes that the quantities between which it stands are to be multiplied together; but very often a full point is used instead of ×, or, still more commonly, one quantity is placed close after the other without any sign between them.

Thus $a \times b$, $a \cdot b$, and ab mean all the same thing, viz. a multiplied by b; and, therefore, if $a = 3$ and $b = 4$, we shall have $ab = 12$, $5a = 15$, $5ab = 60$; and if $c = 5$,

$$\begin{aligned} 4ab + 3ac - 2b + 2abc &= 48 + 45 - 8 + 120 \\ &= 213 - 8 = 205. \end{aligned}$$

7. The number, whether positive or negative, prefixed to any algebraical quantity is called its *coefficient;* thus 3 is the coefficient of $3a$, -7 of $-7ax$, &c.

If no number is expressed, the coefficient is understood, being 1, since a means *once a*.

EXAMPLES.

If $a = 6$, $b = 5$, $c = 4$, $d = 3$, $e = 2$, $f = 1$, and $g = 0$, find the numerical values of the following expressions:

1. $a + 2b + 3c + 4d$.
2. $2a + b - 3c + 4d - 5f + 6g$.
3. $3b - 4a - 6c + 7d + 2e + f - 4g$.
4. $4ag - 3bf + 4ce - d$.
5. $ab + 5bc - 4cde + 5fg$.
6. $-3ab - 2ac + 4bc - abc$.
7. $5ab + 7bd - 8ac + 15cde - 14aef$.

8. $33ab + 7ef + 19cd + 22abg - 13cdef.$
9. $abcd - 2bcde + 3cdef - 4defg.$
10. $3abd + 4bcd - 5abg + 3aefg - 2abcde.$

---

8. The sign ÷ (*by*) denotes that the quantity which stands *before* it is to be divided by that which *follows* it; but, most frequently, to express division, the quantity to be divided is placed over the other with a line between them, in the form of a fraction.

Thus $a \div b$ and $\frac{a}{b}$ denote, either of them, *a divided by b*; and if $a = 2$, $b = 3$, then

$$\frac{5a}{2b} = \frac{10}{6} = \frac{5}{3}, \quad \frac{3a+2b}{2b-a} = \frac{6+6}{6-2} = \frac{12}{4} = 3.$$

9. When any quantity is multiplied by *itself* any number of times, the product is called a *power* of the quantity, and is briefly expressed by writing down the quantity, with a small figure above it to the right, denoting the number of times it is repeated: thus,

a^5 stands for $a \times a \times a \times a \times a$,
$3a^4b^3c^2d$ for $3aaaabbbccd$.

The small figure in any case is called the *index* of the corresponding power; thus,

a (which means a^1) is the *first* power of a,
a^2 . . . . the *second* . . . or *square* of a,
a^3 . . . . the *third* . . . . or *cube* of a,
a^4 . . . . the *fourth* power of a,
&c. &c.

and the small figures, ^{2}, ^{3}, ^{4}, &c. are the indices of the second, third, fourth, &c. powers of a respectively.

Hence, if $a = 2$, $a^4 = 2 \times 2 \times 2 \times 2 = 16$,
if $a = 3$, $a^3 = 3 \times 3 \times 3 = 27$,
if $a = 1$, $a^2 = 1$, $a^3 = 1$, $a^4 = 1$, &c.

EXAMPLES.

If $a = 1$, $b = 3$, $c = 5$, and $d = 0$, find the values of

11. $\dfrac{2b}{a} + \dfrac{3c}{b} + \dfrac{5a}{c} - \dfrac{2a+b}{c}$.

12. $\dfrac{3a+2b}{b} - \dfrac{2b+3c}{7a} + \dfrac{2ab-c}{a}$.

13. $\dfrac{ab+2bc+3cd}{2a+3b} - \dfrac{2abc-4ad+3ac}{3ab-2ad}$
$+ \dfrac{3abc+6ac+6ab-3bc}{6c-2b}$.

14. $a^2 + 2b^2 + 3c^2 + 4d^2$.

15. $3a^2b + 2b^2c - a^2c$.

16. $a^3 - 3a^2c + 3ac^2 - c^3$.

17. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

18. $4abc^2 - 3a^2bc + \dfrac{2ab^2c}{2a+b+c}$.

19. $\dfrac{12a^3-b^2}{3a^2} + \dfrac{2c^2}{a+b^2} - \dfrac{a+b^2+c^3}{5b^3}$.

20. $\dfrac{a^2b^2+1}{a^2+b^2} - \dfrac{1+a^2c^2}{a^2+c^2} + \dfrac{4a+b^2+b^2c^2}{b^2+c^2}$.

---

10. The *square root* of a quantity is that quantity whose square or second power is equal to the given quantity. Thus the square root of 9 is 3, since $3^2 = 9$; the square root of a^2 is a, of 64 is 8.

So also the *cube*, *fourth*, &c. root of a quantity is that quantity whose cube, fourth, &c. power is equal to the given one.

The symbol used to denote a root is $\sqrt{\ }$, (a corruption of r, the first letter of the word *radix*,) which, with the proper index on the left side of it, is set before the quantity whose root is expressed: thus,

$$\sqrt[2]{a^2}=a,\quad \sqrt[3]{64}=4,\quad \sqrt[5]{3125}=5,$$
$$\sqrt[2]{1}=1,\quad \sqrt[3]{1}=1,\ \text{\&c.}$$

The index, however, is generally omitted in denoting the square root; thus $\sqrt{x}$ is written instead of $\sqrt[2]{x}$.

EXAMPLES.

Find the values of

21. $\sqrt{4}+2\sqrt{25}+3\sqrt{49}-\sqrt{64}+3\sqrt{16}$.
22. $\sqrt[3]{8}+2\sqrt[3]{125}-4\sqrt[3]{1}+\sqrt[3]{64}$.
23. $\sqrt[4]{1}+3\sqrt[4]{16}-2\sqrt[5]{32}+3\sqrt[6]{1}$.

If $a=25$, $b=9$, $c=4$, $d=1$, find the values of

24. $\sqrt{a}+2\sqrt{b}+3\sqrt{c}+4\sqrt{d}$.
25. $\sqrt{4a}+\sqrt{9b}+\sqrt{16c}-\sqrt{25d}$.
26. $3\sqrt{a}+2\sqrt{4b}-4\sqrt{9c}+\sqrt{16d}$.
27. $\sqrt[3]{5a}+2\sqrt[3]{3b}-\sqrt[3]{2c}+4\sqrt[3]{d}$.
28. $\sqrt{a^2}+2\sqrt[3]{b^3}+8\sqrt[4]{c^4d^2}-4\sqrt[5]{d}$.
29. $2\sqrt{bc}+3\sqrt{acd}-4\sqrt{b^2d}+3\sqrt{c^2d^3}$.
30. $3\sqrt[3]{2cd}+2\sqrt[4]{9bd}-\sqrt[6]{8cd}+4\sqrt[3]{4c^2}$.

---

11. Those parts of an expression which are connected by + or −, are called its *terms*, and the expression itself is said to be *simple* or *compound*, according as it contains one or more terms.

Thus a^2 and $-b^3$ are each simple quantities, and a^2+ab-b^2 is a compound quantity, whose terms are a^2, $+ab$, and $-b^2$.

12. *Brackets*, ( ), { }, [ ], are employed to shew that all the quantities within them are to be treated as though forming but one quantity. It is of great importance to notice carefully the effect of using them.

Thus $a-(b-c)$ is not the same as $a-b-c$; for, in this last, both b and c are subtracted, whereas in the former it is the compound quantity, $b-c$, which is subtracted.

Hence, if $a=4$, $b=3$, $c=1$, we have

$$a-b-c=4-3-1=0,$$
$$a-(b-c)=4-2=2:$$
$$2a-3b+2c=8-9+2=1,$$
$$2a-(3b+2c)=8-11=-3:$$
$$2a+b-c=8+3-1=10,$$
$$2(a+b)-c=14-1=13,$$
$$2(a+b-c)=12.$$

Sometimes, instead of brackets, a line is used, called a *vinculum*, and drawn above the quantities that are connected; thus $a-\overline{b-c}$ is the same as $a-(b-c)$.

The line, which separates the numerator and denominator of a fraction, may be considered as a species of vinculum; thus $\frac{a+b-c}{4}$ implies that the *whole* quantity $a+b-c$ is to be divided by 4.

13. *Like* algebraical quantities are such as contain the same combinations of letters; thus a and $5a$, $-5a^3b$ $7a^3b$, $3a^2bc$ and $-a^2bc$ are pairs of like quantities.

Unlike quantities are such as contain different combinations of letters; as a^3 and a^2, $3ab$ and $-7a$, $3a^2b$ and $3ab^2$.

EXAMPLES.

If $a=0$, $b=2$, $c=4$, $d=6$, find the values of

31. $3a+(2b-c)^2+\{c^2-(2a+3b)\}$.

32. $3b+(2c-d)^2+\{3b-(2c-d)\}^2-\{3b-(2c-d)^2\}$.

33. $2\sqrt{d-b}+3\sqrt{3d+2c-1}+4\sqrt{a+b+2c+d}$.

34. $3\sqrt[3]{2b^2-a}+2\sqrt[3]{b^2+c^2+7}-\sqrt[3]{(b+c)^2-\left(\frac{d}{2}\right)^2}$.

35. $\{a+(b+c)^2-d\}\,\{(a+b)^2+(d-c)^2\}\,\{(a+b+c)^2-d\}$.

If $a=1$, $b=2$, $c=3$, $d=4$, shew that the numerical values are equal

36. Of $(b+c+d)(b+c-d)(b+d-c)(c+d-b)$ and of $4b^2c^2-\{d^2-(b^2+c^2)\}^2$.

37. Of $\{d-(c-b+a)\}\,\{(d+c)-(b+a)\}$, and of $d^2-(c^2+b^2)+a^2+2(bc-ad)$.

38. Of $\{(b+c)-(d-a)\}^2+\{(c+d)-(b-a)\}^2+\{(b+d)-(c-a)\}^2+(b+c+d-a)^2$, and of $4(a^2+b^2+c^2+d^2)$.

39. Of $\{(a+d)-(c-b)\}\,\{(a+c+d)-b\}\times\{c-(d-a-b)\}\,\{b+c+d-a\}$, and of $4(ad+bc)^2-\{(a^2+d^2)-(b^2+c^2)\}^2$.

40. Of $d^2-(2d-c)\,c+\{2(d-c)+b\}\,b-\{2(d-c+b)-a\}\,a$, and of $\{(d-a)-(c-b)\}^2$.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION.

14. To add *like* algebraical quantities, add separately the positive and negative coefficients; take the difference of these two sums, prefix the sign of the greater, and annex the common letters.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$3a$	$-12bc$	$2c^2$	$3a^2+2b^2$
$-5a$	$4bc$	$-5c^2$	$4a^2-3b^2$
$-2a$	$3bc$	$-7c^2$	$-8a^2+4b^2$
$5a$	$-8bc$	$10c^2$	$5a^2-6b^2$
$6a$	$5bc$	$4c^2$	$7a^2+3b^2$
$7a$	$-8bc$	$4c^2$	$11a^2 \quad *$

In the last Example the star is used to indicate that the terms involving b^2 destroy one another.

If the quantities are *unlike*, we must add any that are like by the preceding rule, and write down the others with their proper signs.

Ex. 5.	Ex. 6.	Ex. 7.
$2a+3b-4c$	$x-2y+3z$	$2a+c+d$
$-3a+4b-c$	$-2x+3y-4z$	$-b+a+e$
$4a+7b+7c$	$3x-5y-5z$	$+c-d$
$a-b-4c$	$x+y$	$-3a-e-f$
$-5a+2b-6c$	$2y+2z$	$-2c+2d-2e$
$-a+15b-8c$	$3x-y-4z$	$-b+2d-2e-f$

EXAMPLES.

Find the sum of

41. $7a-3b+4c-2d+7$, $-8a+4b-6c+2d-11$, $13a+3b-5c+4d-4$, $2a-b+c+11$, $a+2d-3$.

42. $2x - 3y + 4z - 4$, $x + 2y - 3z$, $-3x + 2y - 5z + 7$, $4x - y + 2z - 3$, $9x - 10y + 11z - 12$, $x + y + z$.

43. $2a^2 + ab + 3b^2$, $3a^2 - 4ab + 2b^2$, $3a^2 + 3ab - b^2$, $12a^2 - 14ab - 7b^2$, $3a^2 - 12ab + 17b^2$.

44. $ax - 4by + 3cz$, $13ax - 9by + 7cz$, $-5ax + 7by - 14cz$, $2ax - by + cz$, $-11ax + 13by - 4cz$.

45. $20x^3 + 20x^2y - 3xy^2 + 14y^3$, $-17x^3 + 14x^2y - 12xy^2 - 3y^3$, $14x^3 + 17x^2y + 15xy^2 - 5y^3$, $-12x^3 - 13x^2y - 14xy^2 - 5y^3$, $12x^2y + 3y^3$.

46. $2x^2 - 3xy - 4y^2$, $3xz + 2y^2 - z^2$, $x^2 - 2yz + 5z^2$, $3xy - 6xz - 3x^2$, $3xz - 2z^2 + 5yz$, $4y^2 - 3yz + 2x^2$.

47. $x^3 - 3ax^2 + 3a^2x - a^3$, $4x^3 - 5ax^2 + 6a^2x - 15a^3$, $3x^3 + 4ax^2 + 2a^2x + 6a^3$, $-17x^3 + 19ax^2 - 15a^2x + 8a^3$, $-13ax^2 - 27a^2x + 18a^3$.

48. $a^3 - 2ab^2 - ac^2 + a^2b + 2a^2c + 2abc$, $-a^2b + b^3 - 2bc^2 + 2ab^2 + 2abc + b^2c$, $-2a^2c - b^2c + c^3 + 2abc + ac^2 + 2bc^2$.

49. $3x^3 + 2y^3 + z^3 + 8yz^2$, $y^3 + 3x^2y + 2xy^2 + z^3 - 23x^2z$, $x^3 + 2xyz + 4x^2y + 12x^2z - 9y^2z + 6yz^2$, $2x^3 - 3y^3 + 4xyz - 6xy^2$, $4y^3 - z^3 + 5x^2z - 15xyz + 3y^2z - 14yz^2$, $6x^2z - 15xyz + 4xy^2 - 7x^2y + 6y^2z$.

50. $x^4 + 3xy^3 - xz^3 + x^3y + x^3z$, $3x^2y^2 + 3x^2z^2 + 3xy^2z - 3xyz^2 - 6x^2yz$, $-x^3y + y^4 - yz^3 - 3x^2y^2 + 3x^2yz$, $-3xy^3 - 3xyz^2 - 3y^3z + 3y^2z^2 - 6xy^2z$, $-x^3z + 3y^3z + z^4 + 3x^2yz - 3x^2z^2$, $3xy^2z + xz^3 - 3y^2z^2 + yz^3 + 6xyz^2$.

---

15. To subtract algebraical quantities, change their signs and proceed as in addition.

Thus, taking b from a, we shall have the result $a - b$; since the $-$ prefixed to b shews that it is to be subtracted: and $a - b$ is what would be obtained by *adding* a and $-b$ or, as it is called, is the *sum* of a and $-b$.

Again taking $-b$ from a, we shall have the result $a+b$: for $a=a+b-b$; take away then $-b$ from each of these equal quantities, and the differences will be equal; *i. e.* $a-(-b)=a+b$, which, again, is the *sum* of a and $+b$.

So if we subtract $b-c+d$ from a, the result is $a-b+c-d$.

In many cases, indeed, it will be unnecessary to apply this rule as to the change of sign; since the difference will be often obtained at once, by taking that of the coefficients and annexing the common letters.

Ex. 1.

$$\begin{array}{rl} \text{From} & 5x+7y-8z \\ \text{take} & 3x+y-4z \\ \hline \textit{Ans.} & 2x+6y-4z \end{array}$$

Ex. 2. Ex. 3.

$$\begin{array}{rlcl} \text{From} & 5x^2-2xy+3y^2 & \quad & -3a^2+4ab-5b^2 \\ \text{take} & -4x^2-2xy+7y^2 & & -7a^2+3b^2-2c^2 \\ \hline \textit{Ans.} & 9x^2 \qquad\quad -4y^2 & & 4a^2+4ab-8b^2+2c^2. \end{array}$$

EXAMPLES.

51. From $2a-2b+c$ take $a+b-2c$.

52. From $2x^2-3xy+y^2$ take $4x^2+4xy-2y^2$.

53. From $5ax-7by+cz$ take $ax+2by-cz$.

54. From $7x^2-2x+4$ take $2x^2+3x-1$.

55. From $8a^2-2a+6b^2-5ab+5c^2-3bc+2$
take $a^2+a+2b^2+2ab+3c^2+3bc+2$

56. From $2x^3-4x^2y+6-3xy^2-14y^3$
take $3x^3+2x^2y-3xy^2-10y^3$.

57. From $5x^2+6xy-4y^2-12xz-7yz-5z^2$
take $2x^2-3y^2+4xz-5z^2+6yz-7xy$.

58. From $3x^2 + 2xy - y^2$ take $-x^2 - 3xy + 3y^2$, and from the remainder take $3x^2 + 4xy - 5y^2$.

59. From $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 12b^5$
take $a^5 - 2a^4b - 3a^3b^2$,
and $2a^4b - 4a^3b^2 - 6a^2b^3$,
and $3a^3b^2 - 6a^2b^3 - 9ab^4$,
and $4a^2b^3 - 8ab^4 - 12b^5$.

60. From $a^4 - 2a^3b + 3a^2b^2 - 4ab^3 + 5b^4$
take $2ab^3 - 3a^2b^2 + 4a^3b - 5a^4$
and $3a^4 - 2a^3b + 6a^2b^2 - 2ab^3 + 3b^4$.

---

16. Since the sign + or −, preceding a bracket, will imply (12) that the whole included quantity is to be added or subtracted, if we wish to remove the bracket, we must actually perform the operation indicated by means of it, *i. e* we must add or subtract the quantity in question. We may do so at once in the case of + preceding it, by writing down the included terms with their proper signs, because, when a quantity is added, the signs of its terms are not altered; thus

$$+(a + b - c) = a + b - c, \ (a^2 - 2ab - b^2) = a^2 - 2ab - b^2.$$

But in the case of a − preceding it, we must *change the signs* of all the included terms, according to the Rule for Subtraction (15); thus

$$-(a + b - c) = -a - b + c, \ -(a^2 - 2ab - b^2) = -a^2 + 2ab + b^2,$$

$$\begin{aligned} 3a - \{(a - 3c) - (2b - c)\} &= \\ 3a - (a - 3c) + (2b - c) &= \\ 3a - a + 3c + 2b - c &= 2a + 2b + 2c. \end{aligned}$$

The same remark applies also to the case of a fraction with a numerator of more than one term, whenever the line separating its numerator and denominator, and which (12) is a species of vinculum, is removed by any process.

Thus $-\frac{a+b-c}{4} = -\frac{a}{4} - \frac{b}{4} + \frac{c}{4}$; and $-\frac{a-b}{2}$, when multiplied by 2, becomes $-(a-b)$, or $-a+b$.

EXAMPLES.

61. Reduce
$(a^3 - 2a^2c + 3ac^2) - (a^2c - 2a^3 + 2ac^2) + (a^3 - ac^2 - a^2c)$
to its simplest form.

62. Reduce $(2x^2 - 2y^2 - z^2) - (3y^2 + 2x^2 - z^2)$
$-(3z^2 - 2y^2 - x^2)$ to its simplest form.

63. Reduce $(x^3 + ax^2 + a^2x) - (y^3 - by^2 + b^2y) + (z^3 + cz^2 + c^2z)$
$-(x^3 - y^3 + z^3) + (ax^2 + by^2 + cz^2) - (a^2x - b^2y + c^2z)$
to its simplest form.

64. Reduce $a^2 - (b^2 - c^2) - \{b^2 - (c^2 - a^2)\}$
$+\{c^2 - (b^2 - a^2)\}$ to its simplest form.

65. Reduce $\{2a^2 - (3ab - b^2)\} - \{a^2 - (4ab + b^2)\}$
$+\{2b^2 - (a^2 - ab)\}$ to its simplest form.

66. Reduce $\{x^3 + y^3 - (3x^2y + 3xy^2)\}$
$-\{(x^3 - 3x^2y) - (3xy^2 - y^3)\}$ to its simplest form.

67. Reduce $\{2a - (3b + c - 2d)\} - \{(2a - 3b) + (c - 2d)\}$
$+\{2a - (3b + c) - 2d\} - \{(2a - 3b + c) - 2d\}$
to its simplest form.

---

17. To multiply two simple algebraical quantities together, multiply together respectively the coefficients and letters; and then, if the multiplier and multiplicand have the *same* sign, prefix to this product the sign +, if *different* signs, the sign −.

Thus $7a \times 4b = 28ab$; $-2a \times 3c = -6ac$;
$5b \times -2c = -10bc$; $-3a \times -5b = 15ab$.

18. The rule above given for determining the sign of the product, viz. that *like signs produce* + *and unlike* −, may be thus proved:

1. $+a \times +b$ indicates that $+a$ is to be taken *positively* b times; therefore the product of $+a$ and b, that is, $+ab$, is to be taken positively, and, consequently, the sign of such product is not changed;

hence $+a \times +b = +ab$,

and, similarly, $-a \times +b = -ab$:

2. $+a \times -b$ indicates that $+a$ is to be taken *negatively* b times; therefore the product of $+a$ and b, that is, $+ab$, is to be taken negatively, and, consequently, the sign of such product is changed;

hence $+a \times -b = -ab$,

and, similarly, $-a \times -b = +ab$.

19. If several simple quantities are to be multiplied together, we may multiply them together successively by the preceding rule; thus

$$2a \times -3b \times -4c = -6ab \times -4c = 24abc;$$

but it will be shorter to multiply at once together all the coefficients and all the letters, and then prefix to this product the sign + or −, according as the number of *negative* factors is even or odd.

We may also observe, that, when any number of quantities are multiplied together, it is indifferent in what order they are taken; thus $a \times b \times c$ or abc is the same as bac, or bca, &c. It is, however, usual to place them as much as possible in the order of the letters of the alphabet.

20. The powers of a quantity are multiplied together by *adding the indices;* thus

$$a^3 \times a^2 = a^{3+2} = a^5;$$

$$\text{for } a^3 = a\,.\,a\,.\,a,$$

$$a^2 = a\,.\,a;$$

$\therefore\ a^3 \times a^2 = a\,.\,a\,.\,a\,.\,a\,.\,a = a^5$; and so in other cases.

Hence, $\quad -3a^2b \times 4a^3b^2c \times -2a^2b^3c^2 = 24a^7b^6c^3,$

$$2abc \times 3a^2b^2c^2 \times -ab^2c = -6a^4b^5c^4.$$

21. If the multiplier or multiplicand consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken for the complete product of the two quantities.

This process is generally conducted as in the following Examples.

Ex. 1.

$$\begin{array}{l}
3x^2 - 2xy + 4y^2 \\
2a^2x \\
\hline
6a^2x^3 - 4a^2x^2y + 8a^2xy^2
\end{array}$$

Ex. 2.

$$\begin{array}{l}
-2a^2b^2 + 5ab^3 - 7b^4 \\
-4ab \\
\hline
8a^3b^3 - 20a^2b^4 + 28ab^5
\end{array}$$

Ex. 3.

$$\begin{array}{llll}
2x & + y & & \\
x & + 2y & & \\
\hline
2x^2 & + xy & & \\
 & + 4xy & + 2y^2 & \\
\hline
2x^2 & + 5xy & + 2y^2 &
\end{array}$$

Ex. 4.

$$\begin{array}{llll}
x & + y & & \\
x & - y & & \\
\hline
x^2 & + xy & & \\
 & - xy & - y^2 & \\
\hline
x^2 & & - y^2 &
\end{array}$$

EXAMPLES.

68. Multiply ax^2y^3 by bxy; mx^2 by $-nx^3$; $-acx$ by $-2axy$; abc by bc; $-abc$ by $-ac$; x^2y by $-xy^2$.

69. Multiply $x^2 - xy + y^2$ by x, and $a^2 - ax + x^2$ by $-ax$; $x^2 - ax + b$ by $-abx$; $x^3 - 3x^2y + 3xy^2 - y^3$ by xy.

70. Multiply $2a + b$ by $a + 3b$, and $a - b$ by $c - d$.

71. Multiply $3x + 2y$ by $2x + 3y$, and $3ab + 4b^2$ by $2ab - 3b^2$.

72. Multiply $27x^3 + 9x^2y + 3xy^2 + y^3$ by $3x - y$.

73. Multiply $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$ by $a + 2b$, and by $a - 2b$.

74. Multiply $x^2 + 2ax + 3a^2$ by $x^2 - 2ax + a^2$.

75. Multiply $9a^2 - 3ab + b^2 - 6a - 2b + 4$ by $3a + b + 2$.

76. Multiply $x^2 + y^2 + z^2 + xy - xz + yz$ by $x - y + z$.

77. Multiply $a^3 + 2a^2 + 2a + 1$ by $a^3 - 2a^2 + 2a - 1$.

78. Multiply $a^2 + 4b^2 + 9c^2 + 2ab + 3ac - 6bc$ by $a - 2b - 3c$.

79. Multiply $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x^3 - x^2y + xy^2 - y^3$.

80. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^2 + 2ab + b^2$.

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22. The result obtained, in Ex. 4, of the last Article, for the product of $x + y$ and $x - y$, is a very useful one to be remembered, and the student may apply it to shorten the labour of many examples in Multiplication. Thus, in the next, since the product

of $x + 2$ and $x - 2$ is $x^2 - 2^2$ or $x^2 - 4$,
and of $x + 3$ and $x - 3$ is $x^2 - 9$,

instead of multiplying together the *four* quantities there given, he need only find the product of the *two*, $x^2 - 4$ and $x^2 - 9$.

EXAMPLES.

81. Multiply together $x + 2$, $x - 2$, $x + 3$, and $x - 3$.

82. Multiply together $x + a$, $x - a$, $x + 2a$, and $x - 2a$.

83. Multiply together $a+2b$, $a-2b$, $2a+b$, and $2a-b$.

84. Multiply together $x+1$, $x+2$, $x-1$, $x-4$, and $x+4$.

---

23. To divide one *simple* algebraical quantity by another, divide respectively the coefficient and letters of the dividend by those of the divisor; and then, if the two quantities have the same sign, prefix to the quotient thus obtained the sign +, if different, the sign −

$$\text{Thus } 14ab \div 2a = \frac{14ab}{2a} = 7b;$$

$$-12a \div 10c = -\frac{12a}{10c} = -\frac{6a}{5c};$$

$$-3a \div -6c = +\frac{3a}{6c} = \frac{a}{2c}.$$

24. The rule for the sign of the quotient is the same as that given in (17), viz. that *like signs produce* + *and unlike* −; and is clearly derived from it, for if $+a$ *multiplied* by $-b$ produces $-ab$, of course $-ab$ *divided* by $+a$ produces $-b$; and so in the other cases.

25. One power of a quantity is divided by another by subtracting the index of the latter from that of the former;

$$\text{thus } \frac{a^5}{a^3} = a^{5-3} = a^2;$$

$$\text{for } \frac{a^5}{a^3} = \frac{a^3 . a^2}{a^3} = a^2; \quad \text{so } \frac{x^3y^2z^4}{xyz} = x^2yz^3.$$

26. If the dividend contain several terms, while the divisor still consists of only one, each term of the former must be separately divided by the latter.

$$\text{Ex. 1. } \frac{x^3y-3x^2y^3+9y^2}{3xy} = \frac{x^3y}{3xy} - \frac{3x^2y^3}{3xy} + \frac{9y^2}{3xy} = \frac{x^2}{3} - xy^2 + \frac{3y}{x}.$$

Ex. 2. $$\frac{a^2c^2-2abc^2+3ac^3-4b^3c}{-4abc^2} = -\frac{a^2c^2}{4abc^2}+\frac{2abc^2}{4abc^2}-\frac{3ac^3}{4abc^2}$$

$$+\frac{4b^3c}{4abc^2} = -\frac{a}{4b}+\frac{1}{2}-\frac{3c}{4b}+\frac{b^2}{ac}.$$

27. But if the divisor be also a compound quantity, we must follow just the same course as in common Arithmetic; viz.

(1) Place the quantities, as in Division of Arithmetic, arranging the terms of each of them, so that the different powers of some one letter, common to both of them, may follow in order of the magnitude of their indices.

(2) Divide the first term of the dividend by that of the divisor, and set the result in the quotient.

(3) Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

(4) Bring down fresh terms (as may be required) from the dividend, and repeat the whole operation.

Ex. 1.
$$\begin{array}{lllll}
1-x) & 1 & -2x & +x^2 & (1-x \\
 & 1 & -x & & \\ \cline{2-4}
 & & -x & +x^2 & \\
 & & -x & +x^2 & \\ \cline{2-4}
\end{array}$$

Ex. 2.
$$\begin{array}{llllll}
3x-4y) & 6x^3 & -17x^2y & & +16y^3 & (2x^2-3xy-4y^2 \\
 & 6x^3 & -8x^2y & & & \\ \cline{2-3}
 & & -9x^2y & & +16y^3 & \\
 & & -9x^2y & +12xy^2 & & \\ \cline{3-4}
 & & & -12xy^2 & +16y^3 & \\
 & & & -12xy^2 & +16y^3 & \\ \cline{4-5}
\end{array}$$

Ex. 3.
$$\begin{array}{l} a - x)\, a^4 - x^4\, (a^3 + a^2x + ax^2 + x^3 \\ \qquad\quad \underline{a^4 - a^3x} \\ \qquad\qquad\quad a^3x - x^4 \\ \qquad\qquad\quad \underline{a^3x - a^2x^2} \\ \qquad\qquad\qquad\quad a^2x^2 - x^4 \\ \qquad\qquad\qquad\quad \underline{a^2x^2 - ax^3} \\ \qquad\qquad\qquad\qquad\quad ax^3 - x^4 \\ \qquad\qquad\qquad\qquad\quad \underline{ax^3 - x^4} \end{array}$$

N.B. It is worth observing, that any quantity of the form $a^n - x^n$ (where n represents any whole number) is divisible by $a - x$ without remainder; the quotient being similar in form to that of Ex. 3, and having all its terms *positive*: thus

$$\frac{a^5 - x^5}{a - x} = a^4 + a^3x + a^2x^2 + ax^3 + x^4.$$

Also, that $a^n - x^n$ is divisible by $a + x$, when n is *even*,
and $a^n + x^n$ .... when n is *odd*:

but the quotients here, though similar in form, have their terms *alternately positive and negative*; thus

$$\frac{a^4 - x^4}{a + x} = a^3 - a^2x + ax^2 - x^3,$$

$$\frac{a^3 + x^3}{a + x} = a^2 - ax + x^2.$$

If n be *even*, $a^n + x^n$ is not divisible by either.

Ex. 4.
$$\begin{array}{l} a + b)\, a^2 + 2ab + 2b^2\, (a + b \\ \qquad\quad \underline{a^2 + ab} \\ \qquad\qquad\quad ab + 2b^2 \\ \qquad\qquad\quad \underline{ab + b^2} \\ \qquad\qquad\qquad\quad b^2 \end{array}$$

In this last example we have a remainder b^2, which we place, as in Arithmetic, over the divisor in the form of a fraction, $\frac{b^2}{a+b}$, and add it to the quotient, which thus becomes $a + b + \frac{b^2}{a+b}$.

In this and other cases, as in common Arithmetic, this fraction could not be avoided; but if attention is not paid to the *arrangement according to powers*, alluded to above, (27. 1), and that, not only with the dividend and divisor at starting, but also throughout the sum, care being taken in all the remainders to preserve the order of the indices of the chosen letter, there will *always* be a fractional term of this kind, instead of a clear and complete quotient.

EXAMPLES.

85. Divide ab^2c^3 by abc, $-165x^2y^5$ by $-33xy^3$,
$-70abx^3y$ by $2ax^2y$.

86. Divide $6x^2y - 4x^2z + 6xyz$ by $2x$,
$5a^3b^3 - 35a^2b^2c^2 + 20abc^4$ by $-5ab$, and
$a^3x^3y - 3a^2bx^2y + 3ab^2xy^2 - b^3xy^3$ by $abxy$.

87. Divide $2m^2n^2 - 3mn^3 + 4m^3n - n^4$ by $-3mn^3$, and
$-3a^4b + 5a^3b^2 - 6a^2b^3 - ab^4 + 4b^5$ by $-2a^2b^3$.

88. Divide $6a^2 - 16ab + 8b^2$ by $2a - 4b$, and
$6x^2 + 13xy + 6y^2$ by $2x + 3y$.

89. Divide $x^4 - 81y^4$ by $x - 3y$.

90. Divide $6a^2b^2 - ab^3 - 12b^4$ by $3ab + 4b^2$.

91. Divide $a^5 + 32b^5$ by $a + 2b$.

92. Divide $x^{18} - y^{12}$ by $x^3 + y^2$.

93. Divide $x^6 - 2x^5y + 2x^4y^2 - 4x^3y^3 + 8x^2y^4 + 16xy^5$
$- 32y^6$ by $x^2 - 2y^2$.

94. Divide $1 + 6x^5 + 5x^6$ by $1 + 2x + x^2$.
95. Divide $x^4 - 4xy^3 + 3y^4$ by $x^2 - 2xy + y^2$.
96. Divide $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 12b^5$ by $a^2 - 2ab - 3b^2$.
97. Divide $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
98. Divide $x^6 - a^6$ by $x^3 - 2ax^2 + 2a^2x - a^3$.
99. Divide $a^8 + a^6b^2 - a^2b^6 - b^8$ by $a^3 - a^2b + ab^2 - b^3$.
100. Divide $1 + x^3 - 8y^3 + 6xy$ by $1 + x - 2y$.
101. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
102. Divide $x^3 - 8y^3 - 27z^3 - 18xyz$ by $x - 2y - 3z$.
103. Divide $1 + 2x$ by $1 - 3x$ to 6 terms in the quotient.
104. Divide 1 by $1 - 2x + x^2$ to 6 terms.

CHAPTER III.

GREATEST COMMON MEASURE.

28. WHEN one quantity divides another without remainder, it is said to *measure* it, and is called a *measure* of it; thus, 3, a, b, $3a$, ab, a^2, &c. are all measures of $3a^2b$.

A *common* measure of two quantities is one which divides each of them without remainder; thus, a, b, $3a$, $3b$, ab, $3ab$, are all common measures of $3a^2b$ and $15abc$; and their *greatest* common measure, that is, the largest common factor they contain, is $3ab$.

29. It is commonly easy to detect *by inspection*, *i. e.* by looking at the two quantities, their largest common measure, if it consists of only one term; because then it will be found as a factor in every term of each of them: thus $3xy$ will divide every term of $3x^3y - 6xy^3$ and also of $3xy - 9x^2y^2$; it is therefore a *common measure* of them: and since when these quantities are divided by $3xy$, the quotients $x^2 - 2y^2$ and $1 - 3xy$ have plainly no common factor, $3xy$ is their *greatest* common measure.

So $2a^2b$ is the greatest common measure of $6a^2b^2 - 8a^4b$, and a^2c of $2a^2c^2 - 5a^2bc$; and a^2, which is the greatest common measure of $2a^2b$ and a^2c, is plainly therefore the greatest common measure of the two quantities $6a^2b^2 - 8a^4b$ and $2a^2c^2 - 5a^2bc$.

EXAMPLES.

105. Find the greatest common measure of $3x^3$ and $12x^2y$; of $4a^2b^2$ and $-6ab^3$; of $-12x^2y^3z^4$ and $8y^5z^3$.

106. Of $3ax^2 - 2a^2x$ and $a^2x^2 - 3abx$; of $3a^3 + 2a^2b - 5ab^2$ and $2a^2b + 2ab^2$; of $6x^3y - 12x^2y^2 + 3xy^3$ and $4ax^2 + 4axy + 4a^2x$.

---

30. But if the greatest common measure of two quantities be not a *simple* quantity, it cannot generally be thus easily found, but must be obtained by the following method, in which, for the sake of brevity, we use the letters G. C. M. to denote 'greatest common measure.'

Let there be given then two algebraical quantities, of which it is required to find the G. C. M. Arrange them according to powers of some common letter, and divide whichever of them contains the highest index of this letter by the other; or if the highest index happen to be the *same* in each, take either of them for dividend. Take now, as in common Arithmetic, the remainder after this division for divisor, and the preceding divisor for dividend; and so on until there is no remainder: then the last divisor will be the G. C. M. of the two given quantities.

Ex. 1. Find the G. C. M. of $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$.

$$\begin{array}{l}
x^2 - 7x + 10)\ 4x^3 - 25x^2 + 20x + 25\ (4x + 3 \\
\qquad\qquad\quad\ \underline{4x^3 - 28x^2 + 40x} \\
\qquad\qquad\qquad\qquad 3x^2 - 20x + 25 \\
\qquad\qquad\qquad\qquad \underline{3x^2 - 21x + 30} \\
\qquad\qquad\qquad\qquad\qquad\quad x - 5)\ x^2 - 7x + 10\ (x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{x^2 - 5x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x + 10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2x + 10}
\end{array}$$

Ans. $x - 5$.

EXAMPLES.

Find the G. C. M.

107. Of $3x^2 + x - 2$ and $3x^2 + 4x - 4$.
108. Of $6x^2 + 7x - 3$ and $12x^2 + 16x - 3$.
109. Of $9x^2 - 25$ and $9x^2 + 3x - 20$.
110. Of $8x^2 + 14x - 15$ and $8x^3 + 30x^2 + 13x - 30$.
111. Of $4x^2 + 3x - 10$ and $4x^3 + 7x^2 - 3x - 15$.
112. Of $2x^4 + x^3 - 20x^2 - 7x + 24$ and $2x^4 + 3x^3 - 13x^2 - 7x + 15$.

31. In order to prove the truth of this Rule, we must first shew that, *If a quantity* c *be a common measure of* a *and* b, *it will also measure the sum or difference of any multiples of* a *and* b, *as* ma ± nb.

For let c be contained p times in a, and q times in b; therefore $a = pc$, $b = qc$, and $ma \pm nb = mpc \pm nqc = (mp \pm nq)c$; hence c is contained $mp \pm nq$ times in $ma \pm nb$, and therefore c measures $ma \pm nb$.

32. Let then a and b represent the two given quantities; suppose a divided by b, with quotient p and remainder c; b by c, with quotient q, and remainder d; and suppose c, when divided by d, to give a quotient r, but *no* remainder; d will then be, according to the Rule, the G. C. M. of a and b.

$$\begin{array}{l} b)\ a\ (p \\ \quad \underline{pb} \\ \quad c)\ b\ (q \\ \qquad \underline{qc} \\ \qquad d)\ c\ (r \\ \qquad\quad \underline{rd} \end{array}$$

For, by (31),

every C. M. of a and b is a M. of $a - pb$ or c;
∴ the G. C. M. of a and b is a M. of c,
and, consequently, a C. M. of b and c;
again, every C. M. of b and c is a M. of $pb + c$ or a;
∴ the G. C. M. of b and c is a M. of a,
and, consequently, a C. M. of a and b;
hence the
G. C. M. of a and b is the same as the G. C. M. of b and c;
so the
G. C. M. of b and c .... of c and d;
but if d exactly divide c,
d is the G. C. M. of c and d,
and ∴ is the G. C. M. of a and b.

33. If the given quantities, or either of them, in any case, have *simple* factors, as in (29), but *not common to both* of them, these should be struck out before applying the Rule; and the G. C. M. of the resulting quantities, being found as above, will plainly be the same as that of the given ones.

But if the given quantities have a simple factor, *common to both*, though this also should be struck out of both, it must not be neglected; but, being plainly a factor of the G. C. M. of the given quantities, though not of the resulting quotients, it must be multiplied into the G. C. M. of the latter, as found by the Rule.

From the method of proof in (32), it follows that the above remark applies to any one of the pairs of quantities, b and c, c and d, &c. as well as to a and b; so that whenever we convert a remainder, according to the Rule, into a divisor, we should always strike out any quantity which will divide every term of it.

If, moreover, the first term of any such remainder is negative, it is usual, for the sake of neatness, to change the signs of all its terms, which may be considered as equivalent to dividing it by -1. This change of signs will only affect the *sign* of the G. C. M.

Ex. Find the G. C. M. of

$$2x^5 - 8x^4 + 12x^3 - 8x^2 + 2x \text{ and } 3x^5 - 6x^3 + 3x.$$

Here, striking out of the first the factor $2x$ (which is common to all its terms) and of the second the factor $3x$, we reduce the quantities to

$$x^4 - 4x^3 + 6x^2 - 4x + 1 \text{ and } x^4 - 2x^2 + 1\,;$$

but as $2x$ and $3x$ have a common part, x, the G. C. M. of these two quantities must be multiplied by x to produce that of the given quantities.

$$\begin{array}{rl}
x^4 - 2x^2 + 1) & x^4 - 4x^3 + 6x^2 - 4x + 1\ (1 \\
 & \underline{x^4 - 2x^2 + 1} \\
-4x\,| & \underline{-4x^3 + 8x^2 - 4x} \\
 & \qquad\quad x^2 - 2x + 1
\end{array}$$

$$\begin{array}{rl}
x^2 - 2x + 1) & x^4 - 2x^2 + 1\ (x^2 + 2x + 1 \\
 & \underline{x^4 - 2x^3 + x^2} \\
 & \qquad 2x^3 - 3x^2 + 1 \\
 & \qquad \underline{2x^3 - 4x^2 + 2x} \\
 & \qquad\qquad\quad x^2 - 2x + 1 \\
 & \qquad\qquad\quad \underline{x^2 - 2x + 1}
\end{array}$$

In this example, the first remainder is reduced by dividing it by $-4x$; and the G. C. M. of these two quantities being $x^2 - 2x + 1$, that of the two given quantities will be

$$x\,(x^2 - 2x + 1) \text{ or } x^3 - 2x^2 + x.$$

EXAMPLES.

Find the G. C. M.

113. Of $a^3 + x^3$ and $a^2 + 2ax + x^2$.

114. Of $x^2 + x - 2$ and $x^2 - 3x + 2$.

115. Of $2x^3 + 6x^2 + 6x + 2$ and $6x^3 + 6x^2 - 6x - 6$.

116. Of $2y^3 - 10y^2 + 12y$ and $3y^4 - 15y^3 + 24y^2 - 24$.

117. Of $x^3 - 6ax^2 + 12a^2x - 8a^3$ and $x^4 - 4a^2x^2$.

118. Of $2x^3 + 10x^2 + 14x + 6$ and $x^3 + x^2 + 7x + 39$.

119. Of $3x^3 + 3x^2 - 15x + 9$ and $3x^4 + 3x^3 - 21x^2 - 9x$.

120. Of $2a^4 + a^3b - 4a^2b^2 - 3ab^3$ and $4a^4 + a^3b - 2a^2b^2 + ab^3$.

121. Of $3a^6 + 15a^5b - 3a^3b^2 - 15a^2b^3$ and $10a^5 - 30a^4b - 10a^2b^2 + 30ab^3$.

122. Of $x^4 - 2x^3y + 2xy^3 - y^4$ and $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$.

---

34. If we find, at any step of our process, that the first term of the dividend is not *exactly* divisible by the first of the divisor, then, in order to avoid fractions in the quotient, we may multiply the whole dividend by such a simple factor, as will make its first term so divisible.

If we have attended to the direction of (33), this multiplication will not at all affect the G. C. M. of the two quantities; for no part of the simple factor, thus introduced into the dividend, will be common also to the divisor, which we suppose now stripped of any simple factor it may have contained.

Ex. Find the G. C. M. of

$$6x^2y + 4xy^2 - 2y^3 \text{ and } 8x^3 + 4x^2y - 4xy^2.$$

Stripping them of their simple factors, $2y$ and $4x$, and noting that these contain the common factor, 2, we have $$3x^2 + 2xy - y^2 \text{ and } 2x^2 + xy - y^2,$$

and proceed with these quantities as follows:

$$\begin{array}{rll}
 & 3x^2 + 2xy - y^2 & \\
 & \underline{2} & \\
2x^2 + xy - y^2) & 6x^2 + 4xy - 2y^2\ (3 & \\
 & \underline{6x^2 + 3xy - 3y^2} & \\
 & y\,|\,\underline{xy + y^2} & \\
 & \qquad\qquad x + y) & 2x^2 + xy - y^2\ (2x - y \\
 & & \underline{2x^2 + 2xy} \\
 & & \qquad - xy - y^2 \\
 & & \qquad \underline{- xy - y^2}
\end{array}$$

The G. C. M. then will be $2\ (x + y)$; and it is plain that the G. C. M. of

$$2\ (3x^2 + 2xy - y^2) \text{ and } 2x^2 + xy - y^2$$

will be the same as that of

$$3x^2 + 2xy - y^2 \text{ and } 2x^2 + xy - y^2,$$

because the 2 introduced into the first is no factor of the second quantity.

EXAMPLES.

Find the G. C. M.

123. Of $6x^2 + 13x + 6$ and $8x^2 + 6x - 9$.

124. Of $15x^2 - x - 6$ and $9x^2 - 3x - 2$.

125. Of $6x^3 - 6x^2 + 2x - 2$ and $12x^2 - 15x + 3$.

126. Of $x^4 + 6x^3 + 11x^2 + 4x - 4$ and $x^4 + 2x^3 - 5x^2 - 12x - 4$

127. Of $3x^3 - 22x - 15$ and $5x^4 - 17x^3 + 18x$.

128. Of $3x^3 - 3x^2y + xy^2 - y^3$ and $4x^3 - x^2y - 3xy^2$.

129. Of $x^3 - 8x + 3$ and $x^6 + 3x^5 + x + 3$.

130. Of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

131. Of $6x^4 - x^3y - 3x^2y^2 + 3xy^3 - y^4$ and $9x^4 - 3x^3y - 2x^2y^2 + 3xy^3 - y^4$.

132. Of $12x^5 - 12x^3y^2 + 12x^2y^3 - 3xy^4$ and $12x^5 + 8x^4y - 18x^3y^2 - 6x^2y^3 + 4xy^4$.

35. When one quantity *contains* another, as a divisor without remainder, it is said to be a *multiple* of it; and a *common* multiple of two or more quantities is one that contains each of them without remainder: thus, $6x^3y$ is a common multiple of $2x^2$, $3xy$, $6x^3$, &c., and any quantity is a multiple of any of its measures.

Of course, the *least* common multiple (L. C. M.) of two or more quantities is the least quantity that can be formed, so as thus to contain each of them.

36. *To find the* L. C. M. *of two quantities.*

Let a and b represent the two quantities, d their G. C. M.; and let $a = pd$, $b = qd$, so that p and q will have no common factor. Then the least quantity which contains p and q will be pq, and therefore the least which contains pd and qd will be pqd, which is consequently the L. C. M. required of a and b.

Since $pqd = \frac{pd \times qd}{d} = \frac{a \times b}{d}$, it appears that the L. C. M. of a and b may be found by dividing their product by their G. C. M.; or, which is more simple in practice, by dividing either of them by their G. C. M., and multiplying the *result* by the other.

The L. C. M., however, of two or more quantities, is generally formed by inspection, and, with a little practice, there is no difficulty in this: as we have only to set down the factors, which compose them, omitting any that may occur more than once; and the product of these will be the L. C. M. required.

Ex. 1. Find the L. C. M. of $2bx$, $6abxy$, $3acx$.
Here the factors are $2bx$, $3ay$, c; and the L. C. M. is $6abcxy$.

Ex. 2. Find the L. C. M. of $2a^2(a + x)$, $4ax(a - x)$, $6x^3(a + x)$.

Here the factors are $2a^2(a+x)$, $2x(a-x)$, $3x$; and the L. C. M. is $12a^2x^2(a^2-x^2)$.

EXAMPLES.

Find the L. C. M.

133. Of $4a^2bc$ and $6ab^2c$; of $9x^3y$ and $12xy^3$; of axy and $a(xy-y^2)$; of $ab+ad$ and $ab-ad$.

134. Of $8a^4$, $10a^3b$, and $12a^2b^2$; of a^5, $5a^4b$, $10a^3b^2$, $10a^2b^3$, $5ab^4$, and b^5; of $9x^2$, $6ax$, $8a^2$, $36x^3$, $3ax^2$, $50a^2x$, and $24a^3$.

135. Of $2(a+b)$ and $3(a^2-b^2)$; of $4(a^2-a)$ and $6(a^2+a)$; of $6(x^2+xy)$, $8(xy-y^2)$, and $10(x^2-y^2)$.

136. Of $4(a^3-ab^2)$, $12(ab^2+b^3)$, $8(a^3-a^2b)$; and of $6(x^2y+xy^2)$, $9(x^3-xy^2)$, $4(y^3+xy^2)$.

37. *Every common multiple of* a *and* b *is a multiple of their* L. C. M.

For let m be the L. C. M., and M any other common multiple of a and b; and, if possible, let M contain m r times with remainder s, which will of course be less than m; hence we should have

$$M = rm + s, \quad \text{and, therefore,} \quad s = M - rm:$$

but since a and b measure both M and m, they would also (31) measure $M-rm$, or s; i.e. s, which is less than m, would be a common multiple of a and b, contrary to our supposition that m was their *least* common multiple. Hence M will contain m with no remainder, and will therefore be a *multiple* of m.

CHAPTER IV.

FRACTIONS.

ALGEBRAICAL Fractions are for the most part precisely similar both in their nature and treatment to Simple Vulgar Fractions. We now, however, proceed more completely to consider the theory of them, and the remarks here applied to Algebra may, of course, be transferred to Arithmetic.

38. A Fraction is a quantity which represents a part or parts of an unit or whole.

It consists of two members, the *numerator* and *denominator*, the former placed over the latter with a line between them; the fraction, in fact, expressing also, as we shall presently see, the division of the numerator by the denominator, according to the definition in (8).

The denominator shews into how many equal parts the unit is divided, and the numerator the number of those parts that are taken. Thus $\frac{a}{b}$ means that the unit is divided into b equal parts, a of which are taken.

Every integer may clearly be considered as a fraction whose denominator is 1; thus a is $\frac{a}{1}$.

39. To multiply a fraction by any quantity, multiply the numerator by it, and retain the same denominator.

Thus $\frac{a}{b}$ multiplied by x is $\frac{ax}{b}$; for in each of the fractions $\frac{a}{b}$, $\frac{ax}{b}$, the unit is divided into b equal parts, and x times as many of them are taken in the latter case as in the former.

Conversely, since $\frac{ax}{b} \div x = \frac{a}{b}$, a fraction is divided by dividing the numerator.

40. To divide a fraction by any quantity, multiply the denominator by it, and retain the same numerator.

Thus $\frac{a}{b}$ divided by x is $\frac{a}{bx}$; for the unit being divided into x times as many equal parts in $\frac{a}{bx}$ as it is in $\frac{a}{b}$, any one part in the former case is $\frac{1}{x}$th of any one in the latter, and, the same number of parts being taken in each case, the former fraction is consequently $\frac{1}{x}$th of the latter, *i. e.* is equivalent to the latter divided by x.

Conversely, since $\frac{a}{bx} \times x = \frac{a}{b}$, a fraction is multiplied by dividing the denominator.

41. If any quantity be *both* multiplied and divided by the same quantity, its value will, of course, remain unaltered. Hence if the numerator and denominator of a fraction be *both* multiplied or divided by the same quantity, its value will remain unaltered.

$$\text{Thus } \frac{a}{b} = \frac{ax}{bx} = \frac{a^2}{ab} = \&\text{c.}$$

$$\text{and } \frac{a^3b}{a^2bc} = \frac{a}{c} = \frac{ac}{c^2} = \&\text{c.}$$

42. Since $a = \frac{a}{1}$ (38), and, therefore, a divided by b

$= \frac{a}{1} \div b = \frac{a}{b}$ (40), it follows, as stated in (38), that a fraction represents the quotient of the num^r by the den^r.

Hence it is that in Arithmetic, $\frac{3}{4}$ of £1. is the same as $\frac{1}{4}$ of £3. &c.

43. To reduce a whole number to a fraction with a given den^r, multiply the number by the given den^r, and the product will be the num^r of the required fraction: thus a, expressed as a fraction with den^r x, is $\frac{ax}{x}$; or, with den^r $b - c$, is $\frac{ab - ac}{b - c}$.

The truth of this is evident from (41).

44. The signs of all the terms in both the num^r and den^r of a fraction may be changed without altering its value: thus $\frac{x^2 - 2ax - a^2}{3ax - x^2}$ is identical with $\frac{a^2 + 2ax - x^2}{x^2 - 3ax}$.

This follows also from (41), as the process is equivalent to that of multiplying both num^r and den^r by -1.

45. To reduce a fraction to its lowest terms, divide the num^r and den^r by their G. C. M. (41).

Thus $x + 3$ being the G. C. M. of $x^2 + 4x + 3$ and $x^2 + 5x + 6$, we have $\frac{x^2 + 4x + 3}{x^2 + 5x + 6}$ equivalent to $\frac{x + 1}{x + 2}$.

EXAMPLES.

Reduce to their lowest terms:

137. $\frac{14x^2 - 7xy}{10ax - 5ay}$; $\frac{axy + xy^2}{axy}$; $\frac{5a^3b - 15a^2b^2}{20ab^3 + 10a^2b^2}$.

138. $\frac{6x^3 - 18xy^2}{6x^2y - 12xy^2}$; $\frac{4m^2n^2}{2m^2n + 2mn^2}$; $\frac{3a^2b^2c^2}{a^2bc + ab^2c + abc^2}$.

139. $\frac{x^2 - 4x + 3}{x^2 - 2x - 3}$.

140. $\frac{x^2 + 2x - 3}{x^2 + 5x + 6}$.

141. $\dfrac{6a^2 - 13ax + 6x^2}{10a^2 - 9ax - 9x^2}$. 142. $\dfrac{x^2 + x - 12}{x^3 - 5x^2 + 7x - 3}$.

143. $\dfrac{x^3 + 3x^2 - 4}{x^3 - 1}$. 144. $\dfrac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}$.

---

46. If the highest index of the num^r be lower than that of the den^r, the fraction may be considered in the light of a *proper* fraction in common Arithmetic; if greater, in that of an *improper* fraction, which may be reduced to a kind of *mixed number*, by dividing the num^r by the den^r, as far as the division is possible, and annexing to the quotient the remainder and divisor in the form of a fraction, as in (27, Ex. 4).

Thus $\dfrac{x^3 - 2}{x - 1} = x^2 + x + 1 - \dfrac{1}{x - 1}$.

Conversely, a *mixed* algebraical expression may be reduced to one fraction, by a process similar to that employed in Arithmetic, for reducing a mixed number to a vulgar fraction.

EXAMPLES.

Shew that

145. $\dfrac{3x^2 + 6x + 5}{x + 4} = 3x - 6 + \dfrac{29}{x + 4}$.

146. $\dfrac{30a^3 - 11a^2x - 38ax^2 + 40x^3}{15a^2 + 17ax - 4x^2} = 2a - 3x + \dfrac{7x^2}{5a - x}$.

147. $\dfrac{2x^4 + 19x^2 + 35}{x^3 - 3x^2 + 7x - 21} = 2x + 6 + \dfrac{23}{x - 3}$.

148. $\dfrac{a^3 + x^3}{a^2 + 2ax + x^2} = a - 2x + \dfrac{3x^2}{a + x}$.

149. $\dfrac{48x^3 - 48x^2 + 16x - 16}{4x^2 - 5x + 1} = 12x + 3 + \dfrac{19}{4x - 1}$.

150. $$\frac{x^4 - 9a^2x^2 + 64a^4}{x^3 - 6ax^2 + 13a^2x - 8a^3} = x + 6a + \frac{14a^2}{x - a}.$$

151. $$\frac{a^2 + b^2}{2ab} + 1 = \frac{(a + b)^2}{2ab}; \quad \frac{c^2 + d^2}{2cd} - 1 = \frac{(c - d)^2}{2cd}.$$

152. $$1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a + b)^2 - c^2}{2ab};$$
$$1 - \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2 - (a - b)^2}{2ab}.$$

153. $$x^2 - 3x - \frac{3x(3 - x)}{x - 2} = \frac{x(x^2 - 2x - 3)}{x - 2}.$$

154. $$a^2 - b^2 + \frac{2(a^4 + b^4)}{a^2 + b^2} = \frac{3a^4 + b^4}{a^2 + b^2}.$$

155. $$a^2 + ax + x^2 + \frac{2x^3}{a - x} = \frac{a^3 + x^3}{a - x}.$$

156. $$x - a + y + \frac{a^2 - ay + y^2}{x + a} = \frac{x^2 + xy + y^2}{x + a}.$$

47. To reduce fractions to a common den^{r}, multiply the num^{r} of each fraction by all the den^{rs} except its own, for the new num^{r} corresponding to that fraction; and all the den^{rs} together for the common den^{r}.

The truth of this rule is evident; since, the num^{r} and den^{r} of each fraction being *both* multiplied by the same quantities, viz. the den^{rs} of the other fractions, its value is not altered, though all the fractions now appear with the same den^{r}.

Ex. Reduce $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$ to a common den^{r}.

For the num^{rs} $a \times c \times d = acd$
$b \times b \times d = b^2d$
$c \times b \times c = bc^2$

For the den^{r} $b \times c \times d = bcd$;

and the required fractions are $\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, $\frac{bc^2}{bcd}$.

48. If, however, the original den^rs of the fractions have, any of them, common factors, this process will not give them with their *least* common den^r, which, as in common Arithmetic, will be found by forming, as in (36), the L. C. M. of the given den^rs: and the num^r corresponding to any one of the given fractions will be obtained, by multiplying its num^r by that factor, which will make its den^r equal to the L. C. M.

Ex. Reduce $\frac{a}{2bx}$, $\frac{c}{6abxy}$, $\frac{b}{3acx}$ to a common den^r.

Here the L. C. M. of the den^rs being $6abcxy$, the fractions required are $\frac{3a^2cy}{6abcxy}$, $\frac{c^2}{6abcxy}$, $\frac{2b^2y}{6abcxy}$.

EXAMPLES.

Reduce to common den^rs.

157. $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$; $\frac{x^2}{2ab}$, $\frac{y^2}{3ac}$, $\frac{z^2}{4bc}$; $\frac{2x^2y}{3a^3}$, $\frac{3x^3}{4a^2b}$, $\frac{4y^3}{5ab^2}$, $\frac{5xy^2}{6b^3}$.

158. $\frac{4x^2}{3(a+b)}$, $\frac{5y^2}{4(a-b)}$, $\frac{xy}{6(a^2-b^2)}$.

159. $\frac{x^2}{a^2+b^2}$ and $\frac{y^2}{a^2-b^2}$; $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$.

160. $\frac{1}{4a^3(a+x)}$, $\frac{1}{4a^3(a-x)}$, $\frac{1}{2a^2(a^2-x^2)}$.

---

49. To add or subtract fractions, reduce them to common den^rs, and add or subtract the num^rs for a new num^r, retaining the common den^r.

Ex. 1. Add together $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$.

Reducing them to a common den^r, and adding the num^{rs}, we have the sum required, $\frac{bcx + acy + abz}{abc}$.

Ex. 2. Add $\frac{1+x}{1+x+x^2} + \frac{1-x}{1-x+x^2}$.

The den^{rs} having here no common factor, the sum is

$$\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)} = \frac{2}{1+x^2+x^4}.$$

Ex. 3. From $\frac{1+x}{1+x+x^2}$ take $\frac{1-x}{1-x+x^2}$.

Ans. $\frac{(1+x)(1-x+x^2)-(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)} = \frac{2x^3}{1+x^2+x^4}$.

Ex. 4. Find the value of $2 + \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b}$.

Ans. $\frac{2(a^2-b^2)+(a^2+b^2)-(a-b)(a-b)}{a^2-b^2} = \frac{2a^2+2ab-2b^2}{a^2-b^2}$.

EXAMPLES.

Find the value

161. Of $\frac{a+b}{2} + \frac{a-b}{2}$; $\frac{a+b}{2} - \frac{a-b}{2}$; $\frac{a^2}{a-b} - a$.

162. Of $\frac{a}{a+b} + \frac{b}{a-b}$, and $\frac{a}{a-b} - \frac{b}{a+b}$, and $\frac{a-b}{a+b} + \frac{ab}{a^2-b^2}$.

163. Of $\frac{a^2+b^2}{a^2-b^2} + \frac{a-b}{a+b}$, and $\frac{2x^2-2xy+y^2}{x^2-xy} - \frac{x}{x-y}$.

164. Of $\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2}$, and $\frac{a-b}{ab} + \frac{c-a}{ac} + \frac{b-c}{bc}$.

165. Of $\dfrac{1}{2(a-x)}+\dfrac{1}{2(a+x)}+\dfrac{a}{a^2+x^2}$.

166. Of $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{y}{x-y}+\dfrac{x}{x+y}$.

167. Of $1+\dfrac{x-y}{x+y}-\dfrac{y^2}{x^2-y^2}+\dfrac{2xy}{x^2+y^2}$.

168. Of $\dfrac{x+y}{y}-\dfrac{2x}{x+y}-\dfrac{x^2y-x^3}{x^2y-y^3}$.

169. Of $\dfrac{x}{1-x}-\dfrac{x^2}{(1-x)^2}+\dfrac{x^3}{(1-x)^3}$.

170. Of $\dfrac{1}{x^3}+\dfrac{1}{x^2}-\dfrac{1}{x}-\dfrac{1}{(x^2+1)^2}+\dfrac{x-1}{x^2+1}$.

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50. To multiply fractions, multiply the num^{rs} together for a new num^{r}, and the den^{rs} for a new den^{r}.

Suppose $\dfrac{a}{b}$, $\dfrac{c}{d}$ to be two fractions to be multiplied together. Let x stand for $\dfrac{a}{b}$, and y for $\dfrac{c}{d}$; then $a=bx$, $c=dy$;

$$\therefore\ ac=bdxy,\ \text{and}\ xy=\frac{ac}{bd};$$

$$\text{but}\ \ xy=\frac{a}{b}\times\frac{c}{d},$$

$$\text{and}\ \ \frac{ac}{bd}=\frac{a\times c}{b\times d}=\frac{\text{product of num}^{rs}}{\text{product of den}^{rs}}:$$

whence the truth of the rule is manifest for the product of *two* fractions, and may be similarly proved for any number of fractions.

Ex. Multiply $\dfrac{a+b}{a-b}\times\dfrac{c+d}{c-d}\times\dfrac{3}{2}$.

$$\textit{Ans.}\ \ \frac{3(a+b)(c+d)}{2(a-b)(c-d)}=\frac{3ac+3bc+3ad+3bd}{2ac-2bc-2ad+2bd}.$$

51. To divide one fraction by another, invert the divisor and proceed as in Multiplication.

Suppose that we have to divide $\frac{a}{b}$ by $\frac{c}{d}$:

$$\text{let } \frac{a}{b}=x, \quad \frac{c}{d}=y; \quad \therefore\ a=bx, \quad c=dy,$$

$$\text{and } ad=bdx, \quad bc=bdy;$$

$$\therefore \frac{ad}{bc}=\frac{bdx}{bdy}=\frac{x}{y};$$

$$\text{but } \frac{x}{y}=\frac{a}{b}\div\frac{c}{d}, \quad \text{and } \frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c};$$

whence the truth of the rule is manifest.

Ex. Find the value of $\frac{2a+3b}{2a-3b}\div\frac{c-d}{c+d}$.

$$\textit{Ans. } \frac{(2a+3b)(c+d)}{(2a-3b)(c-d)}=\frac{2ac+3bc+2ad+3bd}{2ac-3bc-2ad+3bd}.$$

N.B. In multiplication and division of fractions, it is always advisable, before multiplying out the factors of the new num^r and den^r, to see if some of them do not exist in *both* the num^r and den^r, in which case they may be struck out (41), and the resulting fraction will appear in a more simple form.

Ex. 1. $\frac{a}{bx}\times\frac{cx}{d}=\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Ex. 2. $\frac{5ax}{3cy}\times\frac{xy+y^2}{x^2-xy}=\frac{5a(x+y)}{3c(x-y)}=\frac{5ax+5ay}{3cx-3cy}$.

Ex. 3. $\frac{4ax}{3by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bc+bx}{a^2-ax}=\frac{4x(a+x)}{3y(c-x)}=\frac{4ax+4x^2}{3cy-3xy}$.

Ex. 4. $\frac{x^2+xy}{x-y}\div\frac{x^4-y^4}{(x-y)^2}=\frac{x^2+xy}{x-y}\times\frac{(x-y)^2}{x^4-y^4}=\frac{x}{x^2+y^2}$.

EXAMPLES.

Find the value

171. Of $\frac{ax}{(a-x)^2} \times \frac{a^2-x^2}{ab}$ and $\frac{a^3-x^3}{a^3+x^3} \times \frac{(a+x)^2}{(a-x)^2}$

172. Of $\left(a-\frac{x^2}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right)$ and $\frac{a^2-b^2}{5b} \times \frac{15a^2}{a+b}$.

173. Of $\frac{2a(x^2-y^2)^2}{cx} \times \frac{x^3}{(x-y)(x+y)^2}$.

174. Of $\frac{a+b}{a^2+2b^2} \times \frac{a-b}{a^2-2b^2}$ and $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4}$.

175. Of $\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b}$ and $\frac{x^3+y^3}{x^2-y^2} \div \frac{x^2-xy+y^2}{x-y}$.

176. Of $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \frac{2ab-2b^2}{3}$.

177. Of $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b}$.

52. A *complex* fraction, *i.e.*, one in which the num^r, or den^r, or both of them, are fractions, may be simplified by the Rule for Division of Fractions.

Thus $$\frac{1-\frac{x}{2}}{4x} = \frac{\frac{2-x}{2}}{\frac{4x}{1}} = \frac{2-x}{2} \times \frac{1}{4x} = \frac{2-x}{8x}.$$

It is well to observe, as in this example, that when a complex fraction is put into the form of a $\frac{\text{fraction}}{\text{fraction}}$, the simple expression for it will be found by taking the product of the upper and lower quantities, or *extremes*, for the num^r, and that of the two middle ones, or *means*, for the den^r. Thus,

Ex. 1. $$\frac{2x}{x-\frac{1}{3}} = \frac{\frac{2x}{1}}{\frac{3x-1}{3}} = \frac{6x}{3x-1}.$$

Ex. 2. $$\frac{5-\frac{x}{4}}{x+1\frac{1}{3}} = \frac{\frac{20-x}{4}}{\frac{3x+4}{3}} = \frac{60-3x}{12x+16}.$$

EXAMPLES.

Simplify

178. $$\frac{2-\frac{3x}{2}}{5};\quad \frac{x}{5-\frac{2x}{3}};\quad \frac{3-x}{x+2\frac{1}{2}};\quad \frac{3x+2\frac{1}{3}}{3\frac{1}{2}}.$$

179. $$\frac{x-\frac{5}{2}(3x-2)}{3};\quad \frac{6x-\frac{3}{4}(3+5x)}{2\frac{1}{4}};\quad \frac{2\frac{1}{3}-\frac{x-2}{2}}{\frac{x+1}{3}-4\frac{1}{2}};$$
$$\frac{1\frac{4}{5}-\frac{2}{3}(x+2)}{\frac{3}{10}(x+1)}.$$

180. $$\frac{\frac{1}{1+x}}{1-\frac{1}{1+x}};\quad \frac{\frac{1}{1+x}+\frac{x}{1-x}}{\frac{1}{1-x}-\frac{x}{1+x}};\quad \frac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}.$$

CHAPTER V.

INVOLUTION AND EVOLUTION.

53. THE process of finding the powers of quantities is called the Involution of them. The examples of involution are therefore only examples of multiplication, where the factors are all the same.

It may be noticed, that any *power of a power* of a simple quantity is obtained by multiplying together the indices of the two powers. Thus the cube of x^2, that is $(x^2)^3 = x^6$; for it

$$= x^2 \times x^2 \times x^2 = x^{2+2+2}\ (20) = x^6.$$

$$\text{So}\quad (x^3)^2 = x^6,\quad (a^3)^4 = a^{12},\quad (a^4)^3 = a^{12},$$

$$(2x^3y^2)^2 = 4x^6y^4,\quad (-2xy^2z^3)^3 = -8x^3y^6z^9,$$

$$\left(\frac{-3a^2b}{c^3}\right)^4 = \frac{81a^8b^4}{c^{12}},\ \&\text{c.}$$

It appears from the above, that the *square of the cube* is the same as the *cube of the square* of any quantity, &c.

Hence also $\{(a-b)^3\}^2 = (a-b)^6$, $\{(x+y)^2\}^4 = (x+y)^8$, &c.; and the actual values of $(a-b)^6$, $(x+y)^8$, &c., are obtained by common multiplication.

Ex. 1. Find the squares of $a+b$ and $a-b$.

$$\begin{array}{lll} a + b & & \\ a + b & & \\ \hline a^2 + ab & & \\ \quad\ + ab & + b^2 & \\ \hline a^2 + 2ab & + b^2. & \end{array} \qquad\qquad \begin{array}{lll} a - b & & \\ a - b & & \\ \hline a^2 - ab & & \\ \quad\ - ab & + b^2 & \\ \hline a^2 - 2ab & + b^2. & \end{array}$$

Ex. 2. Find the square of $a+b+c$.

$$\begin{array}{llllll}
a & +b & +c & & & \\
a & +b & +c & & & \\
\hline
a^2 & +ab & & +ac & & \\
 & +ab & +b^2 & & +bc & \\
 & & & +ac & +bc & +c^2 \\
\hline
a^2 & +2ab & +b^2 & +2ac & +2bc & +c^2.
\end{array}$$

Ex. 3. Find the square of $x-\frac{1}{x}$, and the cube of $1+x$.

$$\begin{array}{lll}
x & -\frac{1}{x} & \\
x & -\frac{1}{x} & \\
\hline
x^2 & -1 & \\
 & -1 & +\frac{1}{x^2} \\
\hline
x^2 & -2 & +\frac{1}{x^2}.
\end{array}
\qquad\qquad
\begin{array}{llll}
1 & +x & & \\
1 & +x & & \\
\hline
1 & +x & & \\
 & +x & +x^2 & \\
\hline
1 & +2x & +x^2 & \\
1 & +x & & \\
\hline
1 & +2x & +x^2 & \\
 & +x & +2x^2 & +x^3 \\
\hline
1 & +3x & +3x^2 & +x^3.
\end{array}$$

N.B. The results of Ex. 1 should be remembered, viz. that the square of $a \pm b$ is $a^2 \pm 2ab + b^2$; as by means of them we may, at once, without the trouble of multiplication, form the squares of other binomials: thus

$$(2x+3y)^2=(2x)^2+2\,(2x)\,(3y)+(3y)^2=4x^2+12xy+9y^2;$$
$$(x-2y)^2=x^2-4xy+4y^2.$$

EXAMPLES.

181. Find the values of $(2ab^2)^2$, $(-3a^2b^2c^4)^3$,

$$\left(-\frac{3ab^2}{4c^3}\right)^4,\ \left(\frac{x^2y^3z^4}{2}\right)^5,\ \text{and}\ \left(-\frac{a^2b^4c^3}{3d^2}\right)^6.$$

Find the square

182. Of $2a-3b$, and of $3x+4y$.

183. Of $3-2x+x^2$, and of $1-x+x^3$.

184. Of $1-x+x^2-x^3$. 185. Of $2x-3y+4z$.

186. Of $3ax+2by+cz$.

187. Of $\frac{a}{b}+\frac{b}{a}$, and of $\frac{2x}{3y}-\frac{3y}{2x}$.

188. Of $\frac{3x^2}{y^2}-\frac{2x}{y}+1-\frac{2y}{x}+\frac{3y^2}{x^2}$.

189. Find the cube of $x^2-\frac{1}{x^2}$, and of $x^2+1+\frac{1}{x^2}$.

190. Find the cube of $a^2-2ab+3b^2$.

54. Since the 4th power is the square of the square (53), we may shorten the operation of finding the 4th power of a quantity by squaring its square; and, similarly, to find the 6th, 8th, &c. powers, we should square the cube, 4th, &c.; to find the 5th, we should take the product of the square and cube (20); to find the 7th, of the cube and 4th; and so on.

Ex. Find the 4th power of $a-x$.

$$
\begin{array}{rlllll}
(a-x)^2 = & a^2 & -\,2ax & +\,x^2 & & \\
 & a^2 & -\,2ax & +\,x^2 & & \\
\hline
 & a^4 & -\,2a^3x & +\,a^2x^2 & & \\
 & & -\,2a^3x & +\,4a^2x^2 & -\,2ax^3 & \\
 & & & +\,a^2x^2 & -\,2ax^3 & +\,x^4 \\
\hline
 & a^4 & -\,4a^3x & +\,6a^2x^2 & -\,4ax^3 & +\,x^4.
\end{array}
$$

EXAMPLES.

Find the 4th power

191. Of $x-2$, $x+2$, and $x-\frac{1}{x}$. 192. Of $x^2+2xy+y^2$.

193. Find the 5th power of $1+2x$.

194. Find the 6th power of $\frac{a}{b}-\frac{b}{a}$.

195. Find the 8th power of $x+\frac{1}{x}$.

55. Since like signs, whether both + or both −, give + in multiplication, it follows that the square of a quantity is the same, whether the quantity be taken positively or negatively; thus $(+a)^2$ and $(-a)^2$ are each $=+a^2$, and $(1-x+x^2)^2$ is the same as $\{-(1-x+x^2)\}^2$, or $(-1+x-x^2)^2$.

Hence also, since as in (54) the 4th, 6th, &c. powers of a quantity are the squares of the 2nd, 3rd, &c. powers, it follows that any *even* power of a quantity is the same, whether that quantity be taken positively or negatively; and, moreover, that no *even* power of *any* quantity can be *negative*.

It is plain also that any *odd* powers of a quantity will have the same sign as the quantity itself.

Def. The sign > is used to express the words *is greater than*, and < to express *is less than*.

56. Whatever be the values of a and b, $a^2+b^2>2ab$.

For, since (55) every square quantity is positive,

$$\therefore\ (a-b)^2 \text{ or } a^2-2ab+b^2 \text{ is positive;}$$

the positive part of this expression must therefore be greater than the negative, or $a^2+b^2>2ab$.

---

57. Evolution, or the extraction of roots, is the method of finding a quantity, which, raised to a proposed power, will produce a given quantity.

Since the cube *power* of $a^2=a^6$, therefore the cube *root* of a^6 is a^2; so $\sqrt{a^4}=a^2$, $\sqrt[4]{16a^8b^4}=2a^2b$, &c.; and we may extract any root of a simple quantity by dividing its index, if possible, by the index of the root.

If, however, the index of the quantity cannot be exactly divided by that of the root (as *e.g.* in the 5th root of a^2, the 2 cannot be divided by 5) then we cannot find the root of it; but can only *indicate* that the root *is to be* extracted, by writing down the quantity, and the sign $\sqrt{\ }$ before it, with the index of the root in question; as $\sqrt[5]{a^2}$, $\sqrt[3]{a^4}$. Such quantities are called *surds*, or *irrational quantities*.

58. It appears as in (55), that any even root of a positive quantity will have a double sign ±, but that there can be no *even* root of a negative quantity; and, also, that any odd root of a quantity will have the same sign as the quantity itself.

$$\text{Thus}\quad \sqrt{\frac{16a^2}{9b^2}} = \pm\frac{4a}{3b},\quad \sqrt[3]{-8x^3y^6} = -2xy^2,$$

$$\sqrt[4]{\frac{625a^4y^8}{81x^4}} = \pm\frac{5ay^2}{3x},\quad \sqrt[5]{32a^5y^{10}} = 2ay^2,\ \text{\&c.}$$

Hence, when we have a surd expressing an *odd* root of a negative quantity, we may write the quantity positive under the sign of evolution, and set the negative sign outside; thus $\sqrt[5]{-x^3} = -\sqrt[5]{x^3}$, $\sqrt[3]{a^6} - \sqrt[3]{-b^2} = a^2 + \sqrt[3]{b^2}$. But this cannot be done with an *even* root of a negative quantity, such as $\sqrt{-x^8}$, which must be left as it is, and is called an *impossible* or *imaginary* quantity; the difference between surd and impossible quantities being that the former have *real* values, though we cannot exactly find them, while there *cannot* be a quantity, positive or negative, an even power of which would produce a negative quantity. Still imaginary quantities are employed in some of the higher applications of Algebra; but for the present we shall leave the consideration both of these and of surd quantities.

EXAMPLES.

196. Find the square roots of $4a^2b^4c^6$, $49x^4y^6z^2$, $100a^8b^{12}c^{16}$.

197. Find the square roots of $\frac{9a^2x^4y^6}{25z^2}$, $\frac{49x^2y^4}{64a^2}$, $\frac{25x^6y^{10}}{16a^2b^4}$.

198. Find $\sqrt{\frac{a^4x^4y^2}{4}}$, $\sqrt[3]{-\frac{8a^3y^6}{27x^9}}$, and $\sqrt[3]{\frac{64b^6c^9}{125a^{12}}}$.

199. Find $\sqrt[3]{-\frac{216a^3b^3c^{15}}{343}}$, $\sqrt[4]{\frac{16x^4y^8}{625a^{12}}}$, and $\sqrt[4]{\frac{81a^4b^8c^{12}}{256x^{16}}}$.

200. Find $\sqrt[5]{\frac{32a^5b^{10}}{c^{15}}}$, and $\sqrt[6]{\frac{64x^{12}y^6}{729z^{18}}}$.

59. To find the square root of a compound quantity.

We *know* that the square of $a+b$ is $a^2+2ab+b^2$ (53. Ex. 1); let us see then how from $a^2+2ab+b^2$, we might deduce its square root $a+b$. Let us write down then, as below, the quantity $a^2+2ab+b^2$. Now a, the first term of the root, may be found immediately, by taking the square root of its first term: set a then on the right, and then subtract a^2; we have now remaining $2ab+b^2$, and if we divide $2ab$ by $2a$, we get $+b$, the other term in the root; lastly, if we add this b to the $2a$, multiply the $2a+b$ thus formed by b, and subtract the product, there is no remainder.

$$\begin{array}{rl}
 & a^2+2ab+b^2\ (a+b \\
 & a^2 \\
\cline{2-2}
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\cline{2-2}
\end{array}$$

Now we may follow this plan in any other case, and if we find no remainder, the root will be exactly obtained.

Ex. 1.

$$\begin{array}{rl} & 9x^2+6xy+y^2\,(3x+y \\ & 9x^2 \\ \hline 6x+y) & 6xy+y^2 \\ & 6xy+y^2 \\ \hline \end{array}$$

Ex. 2.

$$\begin{array}{rl} & 16a^2-56ab+49b^2\,(4a-7b \\ & 16a^2 \\ \hline 8a-7b) & -56ab+49b^2 \\ & -56ab+49b^2 \\ \hline \end{array}$$

Ex. 3.

$$\begin{array}{rl} & 4a^2-4ab-b^2\,(2a-b \\ & 4a^2 \\ \hline 4a-b) & -4ab-b^2 \\ & -4ab+b^2 \\ \hline & -2b^2 \end{array}$$

Here we find a remainder $-2b^2$; we conclude, therefore, that $2a-b$ is *not* the exact root of $4a^2-4ab-b^2$, which is a surd, and can only be written $\sqrt{4a^2-4ab-b^2}$.

EXAMPLES.

Find the square roots of

201. $4x^2+4xy+y^2$, and $25a^2-30ab+9b^2$.

202. $49a^2b^2-14a^3b+a^4$, and $25x^4+30x^3y+9x^2y^2$.

203. $16x^2y^2+40xy^2z+25y^2z^2$, and $25a^4b^2c^2+10a^2bc^5+c^8$.

204. $9x^4+1+\dfrac{1}{36x^4}$, and $25x^2y^2-20x^2+\dfrac{4x^2}{y^2}$.

205. $\dfrac{49x^4y^2}{4}-\dfrac{x^3y^3}{2}+\dfrac{x^2y^4}{196}$, and $\dfrac{1}{4}+ab+a^2b^2$.

---

60. If the root consist of more than two terms, a similar process will enable us to find it, as in the following Example; where it will be seen, that the divisor at any step may be obtained by doubling the quantity already found in the root, or (which amounts to the same thing and is more convenient in practice) doubling the last term of the preceding divisor, and then annexing the new term of the root.

$$\begin{array}{rl}
 & 16x^6-24x^5+25x^4-20x^3+10x^2-4x+1(4x^3-3x^2+2x-1 \\
 & 16x^6 \\
\hline
8x^3-3x^2) & -24x^5+25x^4 \\
 & -24x^5+9x^4 \\
\hline
8x^3-6x^2+2x) & 16x^4-20x^3+10x^2 \\
 & 16x^4-12x^3+4x^2 \\
\hline
8x^3-6x^2+4x-1) & -8x^3+6x^2-4x+1 \\
 & -8x^3+6x^2-4x+1 \\
\hline
\end{array}$$

61. The method of extracting the *cube* root of an algebraical quantity is too complex for this work; but as the 4th *power* of a quantity is the square of its square (54), so the 4th root of a quantity is the square root of its square root, and may therefore be found by the preceding rule. Thus, to find the 4th root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$, the square root will be found to be $a^2 + 2ax + x^2$, and the square root of this to be $a + x$, which is therefore the 4th root of the given quantity.

62. It should be noticed as in (58) that all *even* roots must have double signs; thus the square root of $a^2 + 2ab + b^2$ may be $-(a + b)$, that is, $-a - b$, as well as $a + b$: and, in fact, the first term in the root, which we found by taking the square root of a^2, might have been $-a$ as well as a, and by using this we should have obtained also $-b$ for the second.

So in 59. Ex. 1, the root may also be $-3x - y$; in 60, $-4x^3 + 3x^2 - 2x + 1$; and in all these cases we should get the two roots by giving a double sign to the first term in the root, and using it throughout the sum.

EXAMPLES.

Find the square root

206. Of $1 + 4x + 10x^2 + 12x^3 + 9x^4$.

207. Of $9x^4 + 12x^3 + 22x^2 + 12x + 9$.

208. Of $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$

209. Of $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.

210. Of $\frac{x^2}{4} - \frac{xy}{3} + \frac{y^2}{9} + \frac{xz}{4} - \frac{yz}{6} + \frac{z^2}{16}$.

211. Of $\frac{a^2}{x^2} - \frac{4ab}{xy} + \frac{4b^2}{y^2} + \frac{6ac}{xz} - \frac{12bc}{yz} + \frac{9c^2}{z^2}$.

212. Of $\frac{a^2}{b^2} + \frac{2a}{b} + 3 + \frac{2b}{a} + \frac{b^2}{a^2}$.

213. Of $\frac{x^4}{16} + \frac{3x^3}{2y} + \frac{12x^2}{y^2} + \frac{18x}{y^3} + \frac{36}{y^4}$.

214. Of $x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6$.

215. Of $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

216. Of $4 - 12a + 5a^2 + 14a^3 - 11a^4 - 4a^5 + 4a^6$.

Extract the 4th root

217. Of $1 - 4x + 6x^2 - 4x^3 + x^4$.

218. Of $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.

219. Of $\frac{x^4}{y^4} - \frac{8x^2}{y^2} + 24 - \frac{32y^2}{x^2} + \frac{16y^4}{x^4}$.

220. Find the 8th root of

$$a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8.$$

---

63. The following example will explain the method of extracting the square root of numbers.

Ex.

$$\begin{array}{rl}
 & 1\dot{8}6\dot{7}1\dot{0}4\dot{1}\,(4321 \\
 & 16 \\
\cline{2-2}
83) & \;\;267 \\
 & \;\;249 \\
\cline{2-2}
862) & \;\;\;\;1810 \\
 & \;\;\;\;1724 \\
\cline{2-2}
8641) & \;\;\;\;\;\;8641 \\
 & \;\;\;\;\;\;8641 \\
\cline{2-2}
\end{array}$$

We first set a dot over the last figure of the number, and then, proceeding from it, over every second one. The pairs of figures, 18, 67, 10, 41, thus distinguished are called periods; but it is plain that if the number contain an odd number of figures, the first period will contain but one, instead of two. We now consider what square number is next less than the first period. It is 16 in this case, and we set its square root, 4, as the first figure in the root. The rest of the process, from its perfect similarity to the algebraic, will explain itself. It will be noticed that we take down as the remainder at every step the next period.

64. Although we have not given the mode of algebraic evolution of the cube root, we shall here introduce the process for numbers; the explanation of which is given below, but the proof is too cumbrous for this work.

	Ex.	$8\dot{0}67\dot{7}56\dot{8}16\dot{1}$ (4321
		64
123	4800	16677
	369	
	5169	15507
1292	554700	1170568
	2584	
	557284	1114568
12961	55987200	56000161
	12961	
	56000161	56000161

Here we first divide the number into periods, by placing a dot over the last figure, and then over every third figure beginning from it. We take the cube number next less than the first period 80; this is 64, and we set its cube root, 4, as the first figure in the root; we then subtract the 64, and bring down the next period 677. We now triple the first figure of the root, and set it at some distance to the left of the remainder; the result is 12 (there is 123 in the sum, but the 3 will be accounted for by and by); then we multiply this triple by the first figure of the root, and place the product 48 between the 12 and the remainder, adding two cyphers to it.

We now divide the remainder by this 4800, and get the second figure in the root 3, which we also set after the 12, making 123: now we multiply this 123 by 3, the second figure in the root, and set the product 369 under 4800, add them up, and multiply the sum 5169 by the second figure in the root, subtracting the product 15507 from the remainder. The remainder is 1170, to which we bring down the next period 568, and have now to form the two quantities to the left of it. The first is obtained by tripling the last figure 3 of 123 which gives 129 (the final 2 in 1292 will be accounted for when the next figure in the root is found); and the other quantity 5547, is found by adding 9, the square of the second figure in the root, to the two preceding middle lines $\begin{matrix}369\\5169\end{matrix}$. We have now only to add two cyphers, and repeat the whole process described in this paragraph.

65. In extracting a root of a decimal, we must set the first dot on the last figure of the *whole number,* not of the fractional part, and dot every second or third figure on each side of it, adding cyphers to the decimal part, if necessary. Thus in pointing 24.1 for the square root, the first point would be set on the 4, and a cypher be

added to complete the second period, as $2\dot{4}.1\dot{0}$; so we should dot 1347.2134 for the cube root $\dot{1}34\dot{7}.21\dot{3}40\dot{0}$.

EXAMPLES.

Find the square roots

221. Of 177241, 14356521, 5742.6084, 229.704336.

222. Of 120409, 481636, 4334724, 17338896, 693555.84.

223. Of 1094116, 437.6464, 6595651.24, 129208689.

Find the cube roots

224. Of 1022121, 408.8484, 16.353936, 65415744.

225. Of 32768, 262144, 2097152, 16777216.

226. Of 110592, 884.736, 7097888, 56.783104.

227. Of 1601613, 12812904, 102503.232, 820025856.

228. Of 33076.161, 264.609288, .134217728, 2.116874304.

229. Extract to 6 places of decimals the square roots of 2.5, .2, .001, and .005.

230. Extract to 3 places of decimals the cube roots of 2.5, .2, .01, and .05.

CHAPTER VI.

EQUATIONS.

66. WHEN two algebraical quantities are connected by the sign =, the whole expression is called, according to circumstances we are about to explain, an *identity* or an *equation*.

An *identity* is when the expression is true for *any* numerical values of any of the letters involved in it: thus it is *always* true, whatever be the values of x and y, that

$$(x+y)(x-y)=x^2-y^2,\ \frac{x^3+y^3}{x+y}=x^2-xy+y^2;$$

each of these expressions is therefore an identity, and, in fact, is merely the statement of the equivalence of two different *forms* of the *same* algebraical quantity.

An *equation*, however, is the statement of the equality of two *different* algebraical quantities; in which case the equality does not exist for *any*, but only for some few particular values of one or more of the letters contained in it: thus the equation, $x-3=4$, is true only when we give x the value 7, and $x^2=3x-2$ is true only when we give x the values 1 or 2.

We are about to explain the method of finding these particular values which satisfy the simpler kinds of equations.

67. The last letters of the alphabet x, y, z, &c. are usually employed in equations to denote those quantities to which particular values must be assigned in order to satisfy the equation, and are said to be the *unknown quantities*.

An equation is said to be *satisfied* by any value of the unknown quantity which makes the values of the *two* sides of the equation the *same,* including the case when all its terms lie on one side and 0 on the other, as in $x^2 - 3x + 2 = 0$, which is satisfied by 1 or 2, either of which, being put for x, makes the first side $= 0$.

Those values of the unknown quantities by which the equation is satisfied, are called its *roots;* thus 1 and 2 are the roots of the preceding equation, $x^2 - 3x + 2 = 0$; in $x - 3 = 4$, 7 is the root; in $x^3 - 6 = 6x^2 - 11x$, 1, 2, and 3 are the roots, &c.

68. An equation of one unknown quantity is said to be of as many *dimensions* as is denoted by the index of the highest power of the unknown quantity involved in it.

Thus $x - 3 = 4$ is an equation of one dimension, or a *simple* equation; $x^2 = 3x - 2$ is of two dimensions, or a *quadratic* equation; $x^3 - 6 = 6x^2$ is of three dimensions, or a *cubic* equation; $x^4 - 4x = 13$ is of four dimensions, or a *biquadratic* equation; &c. &c.

It may be noticed, in passing, that it can be proved that every equation of one unknown quantity has as many roots as it has dimensions, and no more.

69. We shall here consider only simple and quadratic equations; but the two following remarks apply to all kinds of equations.

1. *Every term of each side of an equation may be multiplied or divided by the same quantity, without destroying the equality expressed by it.*

Thus, if $3x + \frac{5x}{4} = 34$, multiplying every term by 4,

$$12x + 5x = 136, \quad \text{or} \quad 17x = 136;$$

therefore also, dividing each term by 17, $x = \frac{136}{17} = 8$.

Again, if $12x + 6x = 144$, dividing every term by 6,

$$2x + x = 24, \quad \text{or} \quad 3x = 24;$$

hence also, dividing each term by 3, $x = 8$.
So that 8 is the root of each of these two equations.

Hence an equation may be *cleared of fractions*, by multiplying every term by any common multiple of all the den^rs^. If the L. C. M. be employed, the equation will be expressed in most simple terms.

Thus, if $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$, multiplying every term by 12, which is the L. C. M. of 2, 3, 4, we have

$$\frac{12x}{2} + \frac{12x}{3} + \frac{12x}{4} = 156, \quad \text{or} \quad 6x + 4x + 3x = 156;$$

$$\text{hence} \quad 13x = 156, \quad \text{and} \quad x = \frac{156}{13} = 12.$$

2. *A quantity may be transferred from one side of an equation to the other by changing its sign, without destroying the equality expressed by it.*

Thus if $x - a = y + b$, adding $a - b$ to each side of the equation (which, of course, will not destroy the equality) we have

$$x - a + a - b = y + b + a - b, \text{ or } x - b = y + a;$$

where we see the $-a$ has been transferred to the other side with its sign changed to $+$, and so also the $+b$, with its sign changed to $-$.

Hence if the signs of *all* the terms of both sides of an equation be changed, the equality expressed by it will not be destroyed.

Simple Equations of one unknown Quantity.

70. To solve a simple equation of one unknown quantity,

1. Clear it, when necessary, of fractions;

2. Collect all the terms involving the unknown quantity on one side of the equation, and the known quantities on the other, transposing them, when necessary, with change of sign (69.2);

3. Add together the terms of each side, and divide the sum of the known quantities by the sum of the coeff^s of the unknown quantity; and thus the root required will be found.

Ex. 1. $4x + 2 = 3x + 4$.

There being no fractions in this equation, we have only to collect the terms;

$$\therefore\ 4x - 3x = 4 - 2;$$

$$\therefore\ x = 2, \text{ the root of the equation.}$$

Ex. 2. $4x + 5 = 10x - 16$.

Here $10x - 4x = 16 + 5$;

$$\therefore\ 6x = 21, \text{ and } x = \frac{21}{6} = 3\tfrac{3}{6} = 3\tfrac{1}{2}.$$

Ex. 3. $5(x + 1) - 2 = 3(x - 5)$.

Here, removing the brackets, $5x + 5 - 2 = 3x - 15$;

$\therefore$ collecting terms,

$5x - 3x = -15 - 5 + 2$, or $2x = -18$, and $\therefore x = -9$.

EXAMPLES.

231. $4x - 2 = 3x + 3$.

232. $3x + 7 = 9x - 5$.

233. $4x + 9 = 8x - 3$.

234. $3 + 2x = 7 - 5x$.

235. $x = 7 + 15x$.

236. $3(x - 2) + 4 = 4(3 - x)$.

237. $5 - 3(4 - x) + 4(3 - 2x) = 0$.

238. $13x - 21(x - 3) = 10 - 21(3 - x)$.

239. $3(x - 3) - 2(x - 2) + x - 1 = x + 3 + 2(x + 2) + 3(x + 1)$.

240. $2x - 1 - 2(3x - 2) + 3(4x - 3) - 4(5x - 4) = 0$.

Ex. 4. $\frac{x}{2} - \frac{2x}{3} + \frac{3x}{4} = 11 + \frac{x}{8}$.

Here we first clear the equation of fractions, by multiplying every term by 24, the L. C. M. of the den^{rs}, and (observing that in the first fraction $\frac{24}{2} = 12$, in the second, $\frac{24}{3} = 8$, and so in the others) thus we get

$$12x - 8 \times 2x + 6 \times 3x = 264 + 3x,$$

$$\text{or } 12x - 16x + 18x = 264 + 3x;$$

$\therefore$ collecting terms, $12x - 16x + 18x - 3x = 264$,

$$\text{and } 11x = 264, \therefore x = \frac{264}{11} = 24.$$

Ex. 5. $\frac{x+1}{2} + \frac{x+2}{3} = 14 + \frac{5-x}{4}$.

Multiplying by 12, we have

$$6(x+1) + 4(x+2) = 168 + 3(5-x);$$

$$\therefore 6x + 6 + 4x + 8 = 168 + 15 - 3x,$$

and, collecting, $6x + 4x + 3x = 168 + 15 - 6 - 8$,

$$\text{or } 13x = 169, \text{ and } x = \frac{169}{13} = 13.$$

Ex. 6. $\frac{3x^2 + x}{2} - \frac{2x^2 + x}{3} + \frac{x^2 + x}{4} - \frac{3}{20}$

$$= x^2 + \frac{2}{15} + \frac{x^2 + x}{6} - \frac{x^2 + 5x}{12}.$$

Multiplying by 60, the L. C. M. of the den^{rs}, [and observing the remark at the end of (16)], we have

$$90x^2 + 30x - 40x^2 - 20x + 15x^2 + 15x - 9$$

$$= 60x^2 + 8 + 10x^2 + 10x - 5x^2 - 25x;$$

collecting, we find that the terms involving x^2 destroy one another (otherwise the equation would be a quadratic), and have the result

$$30x - 20x + 15x - 10x + 25x = 9 + 8;$$

$$\therefore\ 40x = 17, \text{ and } x = \frac{17}{40}.$$

EXAMPLES.

241. $\frac{2x}{5} + \frac{x-2}{3} = 2x - 7.$ 242. $\frac{2x}{7} + \frac{x-1}{6} = x - 4.$

243. $\frac{9-2x}{2} = \frac{3}{2} - \frac{7x-18}{10}.$ 244. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{13}{12}.$

245. $x + \frac{14-x}{3} = \frac{21-x}{2}.$ 246. $\frac{x}{2} + \frac{x}{3} = x - 7.$

247. $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} - 1.$

248. $\frac{12x-2}{6} - \frac{18-4x}{3} = x + 2.$

249. $\frac{2x+7}{7} - \frac{9x-8}{11} = \frac{x-11}{2}.$

250. $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 2 - \frac{x}{6} + \frac{5x}{12}.$

251. $\frac{x}{6} - 1\frac{2}{3} = 8\frac{3}{5} + 2\left(\frac{3x}{5} - 1\right).$

252. $\frac{3x-1}{2} - \frac{2x-2}{3} = \frac{x-3}{4} - \frac{x-5}{6} + \frac{16}{3}.$

253. $\frac{2x+3}{4} + \frac{4x}{3} = \frac{1}{x} + \frac{6x+2}{3} - \frac{x+1}{6}.$

254. $\frac{7x+20}{8} - \frac{9x+12}{16} = \frac{3x+1}{10} - \frac{29-8x}{20}.$

255. $\frac{x^2-x+4}{3} = \frac{(2x-3)(3x-2)}{18} + \frac{5}{12}.$

256. $1 - \frac{x}{2}\left(1 - \frac{3}{4x}\right) = \frac{2}{3}\left(3 - \frac{5x}{2}\right) + 5\frac{13}{40}.$

Sometimes the common multiple of the den^{rs} may be too large to be conveniently employed. In this case, it will be best to see if two or three of the fractions have not a simple common multiple, and to get rid of them first.

Ex. 7. $\frac{2x+3}{11} - \frac{x-12}{3} + \frac{3x+1}{4} = 6 + \frac{2x+3}{12}$.

Here the L. C. M. of all the den^{rs} would be 132: but as 12 will include three of them, multiplying by it, we get

$$\frac{12(2x+3)}{11} - 4(x-12) + 3(3x+1) = 72 + 2x + 3;$$

$$\therefore \frac{12(2x+3)}{11} - 4x + 48 + 9x + 3 = 72 + 2x + 3;$$

hence, collecting terms,

$$\frac{12(2x+3)}{11} - 4x + 9x - 2x = 72 + 3 - 48 - 3,$$

$$\text{or} \quad \frac{12(2x+3)}{11} + 3x = 24;$$

$$\therefore 12(2x+3) + 33x = 264, \quad \text{or} \quad 24x + 33x = 264 - 36;$$

$$\therefore 57x = 228, \quad \text{and} \quad x = \frac{228}{57} = 4.$$

Ex. 8. $\frac{x+\frac{4}{5}}{7} - \frac{x}{2} = \frac{4}{35x} - \frac{5x}{14}$.

Reducing the complex fractions as in (52), the equation becomes

$$\frac{5x+4}{35} - \frac{x}{2} = \frac{4}{35x} - \frac{5x}{14}:$$

multiplying then by 35, (and observing that in the last fraction $\frac{35}{14} = \frac{5}{2}$) we have

$$5x + 4 - \frac{35x}{2} = \frac{4}{x} - \frac{25x}{2};$$

multiplying again by $2x$, we have

$$10x^2 + 8x - 35x^2 = 8 - 25x^2;$$

and here the terms involving x^2 destroy one another, and we have

$$8x = 8; \quad \therefore\ x = 1.$$

EXAMPLES.

257. $$\frac{2x-3}{12} - \frac{3x-2}{5} = \frac{4x-3}{8} - 3\tfrac{5}{24}.$$

258. $$\frac{2x}{3} - \frac{1 - \frac{x}{2}}{4x} = \frac{x-1}{2} + \frac{x}{6}.$$

259. $$\frac{x - \frac{1}{3}(x-1)}{3} + \frac{31}{36} = \frac{3 - \frac{1}{4}(x-2)}{5}.$$

260. $$\frac{4x-21}{7} + 7\tfrac{5}{6} + \frac{7x-28}{3} = x + 3\tfrac{3}{4} - \frac{9-7x}{8} + \frac{1}{12}.$$

261. $$\frac{11x-13}{25} + \frac{19x+3}{7} - \frac{5x-25\frac{1}{3}}{4} = 28\tfrac{1}{7} - \frac{17x+4}{21}.$$

262. $$\frac{x+1\frac{1}{2}}{3} - \frac{10-x}{3\frac{2}{3}} = \frac{4 - \frac{2x}{3}}{11} - \frac{1}{11}.$$

263. $$\frac{3x - \frac{2}{3}(1+x)}{4} + \frac{1 - \frac{x}{5}}{5\frac{1}{2}} = \frac{2\frac{2}{5} + \frac{x-1}{25}}{2\frac{1}{5}}.$$

264. $$\frac{x - 1\frac{25}{26}}{2} - \frac{2-6x}{13} = x - \frac{5x - \frac{10-3x}{4}}{39}.$$

If any of the den^{rs} contain *compound* factors, (11), it is best, first to get rid of any *simple* factors they may also contain, and then of each compound factor in turn.

Ex. $\dfrac{75-x}{3(x+1)}+\dfrac{80x+21}{5(3x+2)}=5+\dfrac{23}{x+1}$.

Here, multiplying by 15,

$$\frac{375-5x}{x+1}+\frac{240x+63}{3x+2}=75+\frac{345}{x+1};$$

collecting terms with the same den^rs^,

$$\frac{30-5x}{x+1}+\frac{240x+63}{3x+2}=75;$$

multiplying by $x+1$,

$$30-5x+\frac{240x^2+303x+63}{3x+2}=75x+75,$$

$$\text{whence } \frac{240x^2+303x+63}{3x+2}=80x+45;$$

$$\therefore\ 240x^2+303x+63=240x^2+295x+90,$$

$$\text{and } 8x=27; \quad \therefore\ x=3\tfrac{3}{8}.$$

EXAMPLES.

265. $\dfrac{x-3}{x+2}=\dfrac{1}{2}+\dfrac{x-3}{2x-1}$.

266. $\dfrac{2(4x+3)}{x+3}+\dfrac{3}{x+1}=8$.

267. $\dfrac{x+4}{3x+5}+1\tfrac{1}{6}=\dfrac{3x+8}{2x+3}$.

268. $\dfrac{(2x+3)x}{2x+1}+\dfrac{1}{3x}=x+1$.

269. $\dfrac{10x+17}{18}-\dfrac{12x+2}{13x-16}=\dfrac{5x-4}{9}$.

270. $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}$.

271. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$.

272. $$\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52.$$

273. $$\frac{7x+1}{x-1}=\frac{35}{9}\cdot\frac{x+4}{x+2}+3\tfrac{1}{9}.$$

274. $$\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}.$$

275. $$\frac{6x-7\frac{1}{3}}{13-2x}+2x+\frac{1+16x}{24}=4\tfrac{5}{12}-\frac{12\frac{5}{8}-8x}{3}.$$

276. $$\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-\frac{11}{5}}{6}+\frac{1}{105}.$$

Problems producing Simple Equations.

71. We now proceed to *problems*, or questions, the solution of any one of which depends on that of a simple equation. In these the student must be left almost entirely to his own ingenuity, as no *general* rule can be stated for their solution. The only advice that can be given is to read over carefully and consider well the meaning of the question proposed; then it will always appear that some quantity, at present unknown, is required to be found from the *data* furnished by it; put x to represent this quantity, and now set down in algebraical language the statements made in the question, using x whenever this unknown quantity is wanted in it. We shall thus (in the problems we are now considering) arrive at a simple equation, by means of which the value of x may be ascertained.

Ex. 1. What number is that to which if 8 be added, one fourth of the sum is equal to 29?

Let x represent the number required;

adding 8 to it, we have $x + 8$;

one fourth of this is $\frac{1}{4}(x + 8)$,

and this is to equal 29;

we have, therefore, the equation

$$\tfrac{1}{4}(x + 8) = 29, \text{ whence } x = 108.$$

Ex. 2. What number is that, the double of which exceeds its half by 6?

Let x = the number;

then the double of x is $2x$,

the half of x is $\frac{x}{2}$;

$$\text{hence } 2x - \frac{x}{2} = 6, \text{ whence } x = 4.$$

Ex. 3. A cask, which held 270 gallons, was filled with a mixture of brandy, wine, and water. There were 30 gallons of wine in it more than of brandy, and 30 of water more than there were of wine and brandy together. How many were there of each?

Let x = no. of gals. of brandy;

$\therefore x + 30 =$ .... wine,

and $2x + 30 =$ .... wine and brandy together;

$\therefore 2x + 30 + 30$ or $2x + 60$ = gals. of water;

but the whole number of gallons was 270;

$$\therefore x + (x + 30) + (2x + 60) = 270,$$

whence $x = 45$, the no. of gals. of brandy,

$x + 30 = 75$, ... wine,

$2x + 60 = 150$, ... water.

Ex. 4. A fish was caught whose tail weighed 9lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

It is sometimes convenient to take x to represent, not the quantity actually demanded in the question, but some other unknown quantity on which this one depends. It is only experience, however, and practice which can suggest these cases: but this example is one of them.

Let x = weight of body;

$\therefore\ 9 + \frac{x}{2}$ = weight of tail + $\frac{1}{2}$ body

= weight of head;

but the body weighs as much as head and tail;

$\therefore\ x = \left(9 + \frac{x}{2}\right) + 9$, whence $x = 36$, weight of body;

$\therefore\ 9 + \frac{x}{2} = 27$, weight of head;

and the whole fish weighed

$$27 + 36 + 9 = 72 \text{ lbs.}$$

Ex. 5. A gamester at one sitting lost $\frac{1}{5}$ of his money, and then won 10*s.*; at a second he lost $\frac{1}{3}$ of the remainder, and then won 3*s.*; after which he had 3 guineas left. How much money had he at first?

Let x = number of shillings he had at first; having lost $\frac{1}{5}$ of it, he had $\frac{4x}{5}$ remaining; he then won 10*s.*, and had, therefore, $\frac{4x}{5} + 10$ in hand; losing $\frac{1}{3}$ of this, he had $\frac{2}{3}$ of it remaining,

that is, $\frac{2}{3}\left(\frac{4x}{5} + 10\right)$;

and he then wins 3*s.*, and so has

$$\frac{2}{3}\left(\frac{4x}{5} + 10\right) + 3 \text{ shillings,}$$

which is to equal 3 guineas, or 63*s.*;

hence $\frac{2}{3}\left(\frac{4x}{5}+10\right)+3=63$, whence $x = 100s. = £5$.

Ex. 6. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20 and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half the original number.

Let x = the number;

$$\therefore 20-\frac{3x}{8} = 1^{\text{st}} \text{ remainder},$$

$$\text{and } \frac{x}{4}-\frac{5}{11}\left(20-\frac{3x}{8}\right) = 2^{\text{nd}} \text{ remainder};$$

$$\therefore 12\left\{\frac{x}{4}-\frac{5}{11}\left(20-\frac{3x}{8}\right)\right\} = \frac{x}{2}, \text{ by the question};$$

whence $x = 24$.

EXAMPLES.

277. A bookseller sold 10 books at a certain price, and afterwards 15 more at the same rate, and at the latter time received 35*s*. more than at the former: what was the price per book?

278. What number is that, to which if 7 be added, twice the sum will be equal to 32?

279. What two numbers are those, whose sum is 48 and difference 22?

280. Find a number such that if increased by 10, it will become five times as great as its third part.

281. Divide 150 into two parts, so that one of them shall be $\frac{2}{3}$rds of the other.

282. There is a number such that, if 8 be added to its double, the sum will be five times its half. Find it.

283. *A* and *B* began to play with equal sums; *A* won 30*s*., and then 7 times *A*'s money was equal to 13 times *B*'s: what had each at first?

284. A person bought 20 yards of cloth for 10 guineas, for part of which he gave 11*s.* 6*d.* a yard, and for the rest 7*s.* 6*d.* a yard. How many yards of each did he buy?

285. A is twice as old as B; twenty-two years ago he was three times as old. Required A's present age.

286. Divide £64. among three persons, so that the first may have 3 times as much as the second, and the third, one third as much as the first and second together.

287. A had 18*s.* in his purse, and B, when he had paid A $\frac{2}{3}$ of his money, found that he had now remaining $\frac{2}{5}$ of the sum which A now had; what had B at first?

288. A person buys 4 horses, for the second of which he gives £12. more than for the first, for the third £6. more than for the second, and for the fourth £2. more than for the third. The sum paid for all was £230. How much did each cost?

289. A cistern is filled in $20'$ by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less than the third *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

290. A met two beggars, B and C, and having a certain sum in his pocket, gave B $\frac{3}{28}$ of it, and C $\frac{3}{5}$ of the remainder: A had now 20*d.* left; what had he at first?

291. A garrison of 1000 men was victualled for 30 days; after 10 days it was reinforced, and then the provisions were exhausted in 5 days; find the number of men in the reinforcement.

292. A and B have the same income: A lays by $\frac{1}{5}$ of his; but B, spending £80. per annum more than A, at

the end of 4 years finds himself £220. in debt. What did each receive annually?

293. If A does a piece of work in 10 days which A and B can do together in 7 days, how long would B take to do it alone?

294. A person dies worth £14100; some of this he bequeaths to a Charity, and leaves twelve times as much to his eldest son, whose share is half as much again as that of each of his two brothers, and double of that of each of his three sisters; find the amount of these legacies.

295. A and B began to pay their debts. A's money was at first $\frac{2}{3}$ of B's; but after A had paid £1. less than $\frac{2}{3}$ of his money, and B £1. more than $\frac{7}{8}$ of his, it was found that B had only half as much as A had left. What sum had each at first?

296. At the review of an army, the troops were drawn up into a solid mass, 40 deep, when there were just $\frac{1}{4}$th as many men in front as there were spectators. Had the depth, however, been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Find the number of men of which the army consisted.

Simultaneous Equations of one Dimension.

72. If *one* equation contain *two* unknowns, there are an infinite number of pairs of values of these by which it may be satisfied. Thus in $x = 10 - 2y$, if we give *any value* to y, we shall get a corresponding value for x, by which pair of values the equation will of course be satisfied: if e.g. we take $y = 1$, we shall get $x = 10 - 2 = 8$; if $y = 2$, $x = 6$; if $y = 3$, $x = 4$, &c.

One equation then between *two* unknowns admits of an infinite number of solutions; but if we have as many different equations, as there are quantities, the number of solutions will be limited. Thus, while each of the equations $x=10-2y$, $4x+4=3y$, separately considered, is satisfied by an infinite number of pairs of values of x and y, there will only be found *one* pair common to both, viz. $x=2$, $y=4$, which are therefore the roots of the pair of equations, $x=10-2y$, and $4x+4=3y$.

Equations of this kind, which are to be satisfied by the *same* pair or pairs of values of x and y, are called *simultaneous equations.*

If there be *three* unknowns, there must be *three* equations, and so on. Moreover, these equations must all be *different;* thus, if we had the equation $x=10-2y$, it would be of no use to join with it the equation $2x=20-4y$, (which is obtained by simply doubling it) or any other, derived, like this, immediately from it; since this expresses no new relation between x and y, but repeats in another form the same as before.

There are generally given three methods for solving these equations; but the object aimed at is the same in each, viz. to combine the two equations in such a manner as to expel, or, as the phrase is, *eliminate* from the result one quantity, and so get an equation of *one* unknown only.

73. *First method.*—Multiply, when possible, one equation by some number, that may make the coeff. of x or y in it the same as in the other; then adding or subtracting the two equations, according as these equal quantities have different or same signs, these terms will destroy each other, and the elimination will be effected.

Ex. 1.

$$\left.\begin{array}{r} 4x + y = 34 \\ 4y + x = 16 \end{array}\right\} \begin{array}{l} (1), \\ (2). \end{array}$$

Here mult. (2) by 4, $16y + 4x = 64$,

$$\begin{array}{lrl} & \text{but } y + 4x = 34 & (1); \\ \hline \therefore \text{ subtracting, } & 15y \quad = 30; & \therefore y = 2, \end{array}$$

and (2) $x = 16 - 4y = 16 - 8 = 8$.

Ex. 2.

$$\left.\begin{array}{r} 4x - \ y = \ 7 \\ 3x + 4y = 29 \end{array}\right\} \begin{array}{l} (1), \\ (2). \end{array}$$

Here $3x + 4y = 29$,

$$\begin{array}{lrl} \text{and, mult. (1) by 4, } & 16x - 4y = 28; & \\ \hline \therefore \text{ adding, } & 19x \quad = 57; & \therefore x = 3, \end{array}$$

and (1) $y = 4x - 7 = 12 - 7 = 5$.

Sometimes we cannot make the coeff^s equal by multiplying only one of the equations; but shall have to multiply both by some numbers, which it will be easy to perceive in any case.

Ex. 3.

$$\left.\begin{array}{r} 2x + 3y = \ \ 4 \\ 3x - 2y = -7 \end{array}\right\} \begin{array}{l} (1), \\ (2). \end{array}$$

$$\begin{array}{lrl} \text{Mult. (1) by 3, } & 6x + 9y = \ \ 12 & \\ \text{— (2) by 2, } & 6x - 4y = -14 & \\ \hline \text{subtracting, } & 13y = \ \ 26; & \therefore y = 2, \end{array}$$

and (1) $2x = 4 - 3y = 4 - 6 = -2$; $\therefore x = -1$.

74. *Second method.*—Express one of the unknowns in terms of the other by means of one of the equations, and put this value for it in the other.

Ex. 4.

$$\left.\begin{array}{r} 7x + \dfrac{2y + 4}{5} = 16 \\[2ex] 3y - \dfrac{x + 2}{4} = \ \ 8 \end{array}\right\} \begin{array}{l} (1), \\[4ex] (2). \end{array}$$

Reducing, $$\left.\begin{aligned}35x + 2y &= 76\\ 12y - x &= 34\end{aligned}\right\}\quad\begin{aligned}&(1),\\ &(2).\end{aligned}$$

$\therefore$ (2) $x = 12y - 34$, and (1) $35(12y - 34) + 2y = 76$, whence $y = 3$, and $\therefore x = 2$.

75. *Third method.*—Express the *same* quantity in terms of the other in both equations, and put these values equal.

Ex. 5. $$\left.\begin{aligned}5x - \frac{3y+2}{4} &= 13\\ 7y + \frac{x+2}{5} &= 15\end{aligned}\right\}.$$

Reducing, $$\left.\begin{aligned}20x - 3y &= 54\\ 35y + x &= 73\end{aligned}\right\}\quad\begin{aligned}&(1),\\ &(2).\end{aligned}$$

In (1), $x = \dfrac{54 + 3y}{20}$, in (2), $= 73 - 35y$;

$\therefore \dfrac{54 + 3y}{20} = 73 - 35y$, whence $y = 2$, and $x = 3$.

EXAMPLES.

297. $\left.\begin{aligned}2x + 9y &= 11\\ 4x + y &= 5\end{aligned}\right\}.$

298. $\left.\begin{aligned}2x - 9y &= 11\\ 3x - 12y &= 15\end{aligned}\right\}.$

299. $\left.\begin{aligned}2x - y &= 8\\ 2y + x &= 9\end{aligned}\right\}.$

300. $\left.\begin{aligned}12x + 13y - 37 &= 0\\ 17x - 19y - 15 &= 0\end{aligned}\right\}.$

301. $\left.\begin{aligned}5x + 4y &= 58\\ 3x + 7y &= 67\end{aligned}\right\}.$

302. $\left.\begin{aligned}\frac{x}{2} + \frac{y}{3} &= 13\\ \frac{x}{5} + \frac{y}{8} &= 5\end{aligned}\right\}.$

303. $\left.\begin{aligned}\frac{x}{9} + \frac{y}{8} &= 43\\ \frac{x}{8} + \frac{y}{9} &= 42\end{aligned}\right\}.$

304. $\left.\begin{aligned}2x - \frac{y-3}{5} &= 4\\ 3y + \frac{x-2}{3} &= 9\end{aligned}\right\}.$

305. $$\left.\begin{aligned} (x+5)(y+7) &= (x+1)(y-9)+112 \\ 2x+10 &= 3y+1 \end{aligned}\right\}.$$

306. $$\left.\begin{aligned} \frac{2x-y}{4}+1 &= \frac{7+x}{5} \\ \frac{3-4x}{6}+3 &= \frac{5y-7}{2} \end{aligned}\right\}.$$

307. $$\left.\begin{aligned} \frac{2x+3y}{6}+\frac{x}{3} &= 8 \\ \frac{7y-3x}{2}-y &= 11 \end{aligned}\right\},$$

308. $$\left.\begin{aligned} x-\frac{y-2}{7} &= 5 \\ 4y-\frac{x+10}{3} &= 3 \end{aligned}\right\}.$$

309. $$\left.\begin{aligned} 2x-\frac{y+3}{4} &= 7+\frac{3y-2x}{5} \\ 4y-\frac{8-x}{3} &= 24\tfrac{1}{2}-\frac{2y+1}{2} \end{aligned}\right\}.$$

310. $$\left.\begin{aligned} \frac{3x+4y+3}{10}-\frac{2x+7-y}{15} &= 5+\frac{y-8}{5} \\ \frac{9y+5x-8}{12}-\frac{x+y}{4} &= \frac{7x+6}{11} \end{aligned}\right\}.$$

311. $$\left.\begin{aligned} x-\frac{2y-x}{23-x} &= 20-\frac{59-2x}{2} \\ y+\frac{y-3}{x-18} &= 30-\frac{73-3y}{3} \end{aligned}\right\}.$$

312. $$\left.\begin{aligned} \frac{106y}{63}+\frac{x-6y+1}{7} &= \frac{x-3}{9} \\ \frac{x-5y+8}{9} &= \frac{3x-13y}{7}+\frac{55}{63} \end{aligned}\right\}.$$

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76. Simultaneous equations of three unknowns are solved by eliminating one of them by means of any pair of the equations, and then the same one by means of another pair: we shall thus have two equations involving

two unknowns, which may now be solved by the preceding rules.

Similarly for those of more than three unknowns.

Ex.
$$\left.\begin{array}{r} x - 2y + 3z = 2 \\ 2x - 3y + z = 1 \\ 3x - y + 2z = 9 \end{array}\right\} \quad \begin{array}{l} (1), \\ (2), \\ (3). \end{array}$$

$$\begin{array}{rr} \text{From } (1) & 2x - 4y + 6z = 4, \\ (2) & \underline{2x - 3y + z = 1;} \\ \therefore & -y + 5z = 3 \quad (\alpha). \end{array}$$

$$\begin{array}{rr} \text{Again } (1) & 3x - 6y + 9z = 6, \\ (3) & \underline{3x - y + 2z = 9;} \\ \therefore & -5y + 7z = -3 \quad (\beta), \\ \text{but} & \underline{-5y + 25z = 15} \quad (\alpha); \end{array}$$

$$\therefore \quad -18z = -18, \text{ and } z = 1:$$

$$\begin{array}{ll} \text{hence } (\alpha) & y = 5z - 3 = 2, \\ \text{and } (1) & x = 2 + 2y - 3z = 3. \end{array}$$

EXAMPLES.

313. $$\left.\begin{array}{r} 2x + 3y + 4z = 20 \\ 3x + 4y + 5z = 26 \\ 3x + 5y + 6z = 31 \end{array}\right\}.$$

314. $$\left.\begin{array}{r} 5x + 3y = 65 \\ 2y - z = 11 \\ 3x + 4z = 57 \end{array}\right\}.$$

315. $$\left.\begin{array}{l} x + 2y = 7 \\ y + 2z = 2 \\ 3x + 2y = z - 1 \end{array}\right\}.$$

316. $$\left.\begin{array}{r} 3x + 2y - z = 20 \\ 2x + 3y + 6z = 70 \\ x - y + 6z = 41 \end{array}\right\}.$$

317. $$\left.\begin{array}{l} x + y + z = 5 \\ x + y = z - 7 \\ x - 3 = y + z \end{array}\right\}.$$

318. $$\left.\begin{array}{r} 2(x - y) = 3z - 2 \\ x + 1 = 3(y + z) \\ 2x + 3z = 4(1 - y) \end{array}\right\}.$$

319. $$\left.\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} = 12 - \dfrac{z}{6} \\[2ex] \dfrac{y}{2} + \dfrac{z}{3} = 8 + \dfrac{x}{6} \\[2ex] \dfrac{x}{2} + \dfrac{z}{3} = 10 \end{array}\right\}.$$

320. $$\left.\begin{array}{r} y + \dfrac{z}{3} = \dfrac{x}{5} + 5 \\[2ex] \dfrac{x-1}{4} - \dfrac{y-2}{5} = \dfrac{z+3}{10} \\[2ex] x - \dfrac{2y-5}{3} = 1\tfrac{3}{4} - \dfrac{z}{12} \end{array}\right\}.$$

PROBLEMS.

321. What fraction is that to the num^r of which if 7 be added, its value is $\frac{2}{3}$; but if 7 be taken from the den^r its value is $\frac{3}{8}$?

322. A bill of 25 guineas was paid with crowns and half guineas; and twice the number of half guineas exceeded three times that of the crowns by 17: how many were there of each?

323. Two labourers, *A* and *B*, received £5. 17*s*. for their wages, *A* having been employed 15, and *B* 14 days; and *A* received for working four days 11*s*. more than *B* did for three days: what were their daily wages?

324. A farmer parting with his stock sells to one person 9 horses and 7 cows for £300; and, to another, at the same prices, 6 horses and 13 cows for the same sum: what was the price of each?

325. A draper bought two pieces of cloth for £12. 13*s*., one being 8*s*. and the other 9*s*. per yard. He sold them each at an advanced price of 2*s*. per yard, and gained by the whole £3. What were the lengths of the pieces?

326. There is a number of two digits, which, when divided by their sum, gives the quotient 4; but if the digits be inverted, and the number thus formed be increased by 12, and then divided by their sum, the quotient is 8. Find the number.

327. A rectangular bowling-green having been measured, it was observed that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 feet more. Find its present area.

328. The Star leaves Cambridge for London with a certain number of passengers, 4 more outside than inside; and 7 of the former would together pay 2*s*. less than 4 of the latter. The fare of the whole was £6. 4*s*. But they took up, half way, 3 more outsides and one inside, and the whole fare received was now $\frac{37}{31}$ of what it was at first. What was the inside fare, and the whole number of passengers at first?

329. A boy at a fair spends his money in oranges: if he had received 5 more for his money, they would have averaged a halfpenny each less; if 3 less, a halfpenny each more. How much did he spend?

330. A person rows from Cambridge to Ely, a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the times of his going and returning.

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77. *Quadratic Equations* are those in which the *square* of the unknown quantity is found. Of these there are two species;

Pure Quadratics, in which the square only is found, without the first power, as $x^2 - 9 = 0$, &c.;

Adfected Quadratics, where the first power enters as well as the square, as $x^2 - 3x + 2 = 0$, &c.

78. Pure Quadratics are solved, as in simple equations, by collecting the unknown quantities on one side, and the known quantities on the other. We shall thus find the value of x^2, and thence the value of x, to which we must prefix a double sign ($\pm$) (62).

Such equations therefore will have two equal roots, with contrary signs.

Ex. 1. $x^2 - 9 = 0$.

$$\text{Here } x^2 = 9, \quad \therefore x = \pm 3.^*$$

Ex. 2. $\dfrac{3x^2 + 5}{8} - \dfrac{x^2 + 21}{3} = 39 - 5x^2$.

Reducing, $121x^2 = 1089$; $\therefore x^2 = 9$, and $x = \pm 3$.

EXAMPLES.

331. $\dfrac{x^2}{2} = 14 - 3x^2$.

332. $x^2 + 5 = \dfrac{10x^2}{3} - 16$.

333. $(x + 2)^2 = 4x + 5$.

334. $\dfrac{3}{1 + x} + \dfrac{3}{1 - x} = 8$.

335. $8x + 1 + \dfrac{7}{x} = \dfrac{21 + 65x}{7} - 2$.

336. $\dfrac{1}{2x^2} + \dfrac{1}{4x^2} - \dfrac{1}{6x^2} = \dfrac{7}{3}$.

337. $\dfrac{3x^2}{4} - \dfrac{15x^2 + 8}{6} = 2x^2 - 3$.

338. $\dfrac{x^2}{5} - \dfrac{x^2 - 10}{15} = 5\tfrac{3}{4} - \dfrac{50 + x^2}{25} + \dfrac{x^2}{20}$.

339. $\dfrac{10x^2 + 17}{18} - \dfrac{12x^2 + 2}{11x^2 - 8} = \dfrac{5x^2 - 4}{9}$.

340. $\dfrac{14x^2 + 16}{21} - \dfrac{2x^2 + 8}{8x^2 - 11} = \dfrac{2x^2}{3}$.

---

[*] Since the square root of x^2 is $\pm x$, we should have $\pm x = \pm 3$; but of the four values

$$+x = +3, \quad +x = -3, \quad -x = +3, \quad -x = -3,$$

the first and fourth are identical, and so are the second and third: hence there are only two *different* values of x, viz. $x = +3$, or $x = -3$, which are those obtained by prefixing the double sign to *one* of the square roots only.

341. $\frac{45+3x^2}{x^2+3}+\frac{90+4x^2}{x^2+9}=7+\frac{72}{x^2+3}$.

342. $\frac{3x^2+5}{10}-\frac{2x^2-5}{15}+2\frac{3}{4}=\frac{7x^2-20}{20}$.

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79. An *adfected* quadratic may always be reduced to the simple form,

$$x^2+px+q=0,$$

where the coeff. of x^2 is + 1, and p, q, represent numbers or known quantities: and this equation may be solved as follows.

Since $x^2+px+q=0$, $\therefore x^2+px=-q$;

and, adding $\left(\frac{p}{2}\right)^2$ to each side,

$$x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q:$$

now, by this step, the first side becomes a complete square; taking, therefore, the square root of each, and prefixing, as before, the double sign to that of the latter, we have

$$x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}-q}, \text{ and } x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q};$$

which expression gives us, according as we take the upper or lower sign, two roots of the quadratic.

80. From the preceding we derive the following Rule for the solution of an adfected quadratic:

Reduce it to its simplest form; set the terms involving x^2 and x on one side, (the coeff. of x^2 being + 1,) and the known quantity on the other: now, if we *add the square of half the coeff. of* x *to each side,* the first will become a complete square; and taking the square root of each, prefixing the double sign to the second, we shall obtain, as above, the two roots of the equation.

Ex. 1. $x^2 - 6x = 7$.

Half the coeff. of x being 3, we add 3^2, or 9, to each side;
$$\therefore\ x^2 - 6x + 9 = 7 + 9 = 16,$$
$$\text{whence } x - 3 = \pm 4;$$
$$\text{and } x = 3 + 4 = 7, \quad \text{or } x = 3 - 4 = -1;$$
so that 7 and -1 are the two roots of the equation.

Ex. 2. $x^2 + 14x = 95$.

Adding 7^2 or 49 to each side, we have
$$x^2 + 14x + 49 = 95 + 49 = 144,$$
$$\text{whence } x + 7 = \pm 12;$$
$$\text{and } x = -7 + 12 = 5, \quad \text{or } x = -7 - 12 = -19.$$

EXAMPLES.

343. $x^2 - 2x = 8$.

344. $x^2 + 10x = -9$.

345. $x^2 - 14x = 120$.

346. $x^2 - 12x = -35$.

347. $x^2 + 32x = 320$.

348. $x^2 + 100x = 1100$.

If the coeff. of x be *odd*, its half will be a fraction. In adding its square to the *first* side, we may *express* the squaring, without effecting it, by means of a bracket.

Ex. 3. $x^2 - 5x = -6$.

Adding $\left(\frac{5}{2}\right)^2$ or $\frac{25}{4}$ to each side, we have
$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = -6 + \frac{25}{4} = \frac{-24 + 25}{4} = \frac{1}{4};$$
$$\therefore\ x - \frac{5}{2} = \pm\frac{1}{2},$$
$$\text{and } x = \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3, \quad \text{or } x = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2.$$

Ex. 4. $x^2 - x = \frac{3}{4}$.

Here the coefficient of x is 1;

adding, therefore, $\left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$ to each side, we have

$$x^2 - x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1;$$

$$\therefore\ x - \frac{1}{2} = \pm 1,$$

$$\text{and } x = \frac{1}{2} + 1 = 1\tfrac{1}{2}, \quad \text{or } x = \frac{1}{2} - 1 = -\frac{1}{2}.$$

EXAMPLES.

349. $x^2 + 7x = 8$.
350. $x^2 - 13x = 68$.
351. $x^2 + 25x = -100$.
352. $x^2 + 13x = -12$.
353. $x^2 + 19x = 20$.
354. $x^2 + 111x = 3400$.

If the coeff. of x be a fraction, its half will, of course, be found by halving the numʳ, if possible; if not, by doubling the denʳ.

Ex. 5. $x^2 + \frac{10x}{3} = 19$.

$$\text{Here } x^2 + \frac{10x}{3} + \left(\frac{5}{3}\right)^2 = 19 + \frac{25}{9} = \frac{171 + 25}{9} = \frac{196}{9};$$

$$\therefore\ x + \frac{5}{3} = \pm \frac{14}{3},$$

$$\text{and } x = -\frac{5}{3} + \frac{14}{3} = 3, \quad \text{or } x = -\frac{5}{3} - \frac{14}{3} = -6\tfrac{1}{3}.$$

Ex. 6. $x^2 + \frac{13x}{5} = 74$.

$$\text{Here } x^2 + \frac{13x}{5} + \left(\frac{13}{10}\right)^2 = 74 + \frac{169}{100} = \frac{7400 + 169}{100} = \frac{7569}{100};$$

$$\therefore\ x + \frac{13}{10} = \pm\frac{87}{10},$$

and $x = -\frac{13}{10} + \frac{87}{10} = 7\frac{2}{5}$, or $x = -\frac{13}{10} - \frac{87}{10} = -10$.

EXAMPLES.

355. $x^2 - \frac{x}{3} = 34.$

356. $x^2 - \frac{3x}{2} = 27.$

357. $x^2 + \frac{7x}{5} = 86.$

358. $x^2 - \frac{26x}{7} = 144.$

359. $x^2 + \frac{x}{12} = 145.$

360. $x^2 - \frac{22x}{13} = 147.$

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81. In the following Examples the equations will first require reduction; and in order to make the coeff. of x^2 to be $+1$ (as the Rule requires) if it have any other coeff., we must divide each term of the equation by it: thus, if $3x^2 - 20x = 7$, we have $x^2 - \frac{20x}{3} = \frac{7}{3}$.

EXAMPLES.

361. $x = \frac{5}{3} + \frac{x^2}{12}.$

362. $2x = 4 + \frac{6}{x}.$

363. $\frac{7x^2}{11} - \frac{2x}{3} = \frac{11x + 18}{33}.$

364. $\frac{3}{4}(x^2 - 3) = \frac{x - 3}{8}.$

365. $2x^2 + 1 = 11(x + 2).$

366. $11x^2 - 9x = 11\frac{1}{4}.$

367. $\frac{x + 22}{3} - \frac{4}{x} = \frac{9x - 6}{2}.$

368. $\frac{x}{x + 1} + \frac{x + 1}{x} = \frac{13}{6}.$

369. $x - \frac{x^3 - 8}{x^2 + 5} = 2.$

370. $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = 2\frac{1}{3}.$

371. $\frac{2x}{x - 4} + \frac{2x - 5}{x - 3} = 8\frac{1}{3}.$

372. $\frac{1}{x - 1} - \frac{1}{x + 3} = \frac{1}{35}.$

373. $\frac{48}{x + 3} = \frac{165}{x + 10} - 5.$

374. $\frac{x+4}{3} - \frac{7-x}{x-3} = \frac{4x+7}{9} - 1.$

375. $\frac{2x+9}{9} + \frac{4x-3}{4x+3} = 3 + \frac{3x-16}{18}.$

376. $\frac{3x-7}{x} + \frac{4x-10}{x+5} = 3\tfrac{1}{2}.$

377. $\frac{5x}{x+4} - \frac{3x-2}{2x-3} = 2.$ 378. $\frac{4x+7}{19} + \frac{5-x}{3+x} = \frac{4x}{9}.$

---

82. Any equation, in which the unknown quantity is found in two terms, and with its index in one twice as great as in the other, may be solved as a quadratic equation.

Ex. $x^4 - 13x^2 = -36.$

$$\text{Here } x^4 - 13x^2 + \left(\frac{13}{2}\right)^2 = \frac{169}{4} - 36 = \frac{25}{4};$$

$$\therefore\ x^2 - \frac{13}{2} = \pm\frac{5}{2},$$

$$\text{and } x^2 = \frac{13}{2} + \frac{5}{2} = 9, \quad \text{or } x^2 = \frac{13}{2} - \frac{5}{2} = 4,$$

$$\text{whence } x = \pm 3, \text{ or } \pm 2:$$

and thus the given equation, which is a *biquadratic*, admits (68) of the *four* roots, + 2, – 2, + 3, – 3.

EXAMPLES.

379. $x^4 - 5x^2 + 4 = 0.$ 380. $(x^2-2)^2 = \frac{1}{4}(x^2+12).$

381. $(x^2-9)^2 = 3 + 11(x^2-2).$

382. $(x^2-1)(x^2-2) + (x^2-3)(x^2-4) = x^4 + 5.$

383. $(x+1)^3 - x^2(x^2-1) = (x-1)^3 + 2(x^2+3).$

384. $(x^2+x+1)^2 - \frac{x^2+3}{4} = 2x(x+1)^2.$

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83. Since the roots of $x^2+px+q=0$ are $-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$, if $\frac{p^2}{4}>q$, we have $\frac{p^2}{4}-q$ *positive*, and, therefore, $\sqrt{\frac{p^2}{4}-q}$ *rational*; and since, in one root, it is taken with a +, and in the other with a − sign, the two roots will be *real and different* in value: if $\frac{p^2}{4}=q$, we have $\frac{p^2}{4}-q=0$, and, therefore, the two roots will be *real and equal* in value; if $\frac{p^2}{4}<q$, we have $\frac{p^2}{4}-q$ *negative*, and, therefore, $\sqrt{\frac{p^2}{4}-q}$ *impossible*, and the two roots will be *impossible*.

Hence if any equation be expressed in the form $x^2+px+q=0$, its roots will be *real and different*, *real and equal*, or *impossible*, as $p^2 >, =,$ or $< 4q$.

So also in the more general equation, $ax^2+bx+c=0$, or $x^2+\frac{b}{a}x+\frac{c}{a}=0$, the roots are *real and different*, *real and equal*, or *impossible*, as $\left(\frac{b}{a}\right)^2 >, =,$ or $< 4\frac{c}{a}$, that is, as $b^2 >, =,$ or $< 4ac$: thus

$$\text{in } 6x^2-7x+2=0, \text{ they are } \textit{real and different},$$
$$\because\ 7^2>4\times6\times2, \text{ or } 49>48:$$
$$\text{in } 9x^2-12x+4=0, \text{ they are } \textit{real and equal},$$
$$\because\ 12^2=4\times9\times4, \text{ or } 144=144:$$
$$\text{in } 3x^2-7x+15=0, \text{ they are } \textit{impossible},$$
$$\because\ 7^2<4\times3\times15, \text{ or } 49<180.$$

We may here notice the following properties of quadratic equations.

84. If α, β represent the two roots of $x^2+px+q=0$, then $-p=\alpha+\beta$, and $q=\alpha\beta$.

For, by (79),

$$a = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \quad \beta = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q};$$

$$\therefore\ a + \beta = -p, \quad \text{and} \quad a\beta = \frac{p^2}{4} - \left(\frac{p^2}{4} - q\right) = q.$$

Hence, when any quadratic is reduced to the form $x^2 + px + q = 0$, we have

coeff. of 2nd term, with sign changed, = *sum* of roots,
and 3rd term = *product* . . .

Thus, in 80, Ex. 1, the equation, when expressed in this form, is $x^2 - 6x - 7 = 0$, and the roots are there found, 7 and -1;

and here $+6 = 7 + (-1) =$ *sum* of roots,
and $-7 = 7 \times (-1) =$ *product* . . . :

in 80, Ex. 2, the equation is $x^2 + 14x - 95 = 0$, the roots, 5 and -19;

and here $-14 = 5 + (-19) =$ *sum* of roots,
and $-95 = 5 \times (-19) =$ *product* . . . :

so also $ax^2 + bx + c = 0$, when expressed in this form, becomes $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$;

and here $-\frac{b}{a} =$ *sum* of roots, $\frac{c}{a} =$ *product.*

85. *If* a, β *be the roots of* $x^2 + px + q = 0$, *then*

$$x^2 + px + q = (x - a)(x - \beta).$$

For, by (84), $-p = a + \beta$, and $q = a\beta$,

$$\begin{aligned} \therefore\ x^2 + px + q &= x^2 - (a + \beta)x + a\beta \\ &= x^2 - ax - \beta x + a\beta \\ &= (x - a)(x - \beta). \end{aligned}$$

Hence we may form an equation with any given roots: thus, the equation, whose roots are 2 and 3, is

$$(x - 2)(x - 3) = x^2 - 5x + 6 = 0;$$

the equation, whose roots are -2 and $\frac{1}{4}$, is

$$(x+2)(x-\tfrac{1}{4}) = x^2 + \tfrac{7}{4}x - \tfrac{1}{2} = 0,$$

or, clearing it of fractions, $4x^2 + 7x - 2 = 0$.

This law of formation is not confined to quadratics, but is true of equations of all dimensions; thus the biquadratic whose roots are $-1, 2, -2, 3$, is

$$(x+1)(x-2)(x+2)(x-3) = x^4 - 2x^3 - 7x^2 + 8x + 12 = 0.$$

86. If one of the roots be 0, the corresponding factor will be $x - 0$ or x; thus if the roots are 0, 1, 3, the equation will be

$$x(x-1)(x-3) = x^3 - 4x^2 + 3x = 0.$$

In such a case then x will occur in *every* term of the equation. Now if the above equation were proposed for solution, we may strike an x out of every term, and thus reduce it to $x^2 - 4x + 3 = 0$, which would give us the two roots, 1 and 3: but whenever, as here, we strike an x out of every term of an equation, we must not neglect it; since such an equation, as it originally stood, would plainly be satisfied by $x = 0$, which is therefore one of its roots.

EXAMPLES.

385. Form an equation with roots -3 and 7.
386. Form an equation with roots $\frac{2}{3}$ and $-\frac{3}{2}$.
387. Form an equation with roots $3, -3, \frac{3}{4}, -\frac{3}{4}$.
388. Form the equation with roots 0, 1, 2, 3.
389. Form the equation with roots $0, -1, 2, -2, \frac{1}{4}$.

We shall now give a few examples of quadratic equations of *two* unknowns. In these, however, the elimination of one of them cannot generally be performed as easily as in simple simultaneous equations; but in many cases the solution must be obtained by means and artifices, which practice and experience only can suggest.

There are, however, three cases of frequent occurrence, for which the following observations will be useful.

87. If in one of the equations either of the unknowns can be expressed in terms of the other, this value may be put for it in the other equation, which will now, as in (74), contain only one unknown.

Ex. 1.
$$\left.\begin{aligned} x+\frac{1}{3}&=\frac{2x+y}{3} \qquad (1)\\ \frac{x+y}{x}&=\frac{4x-y}{2} \qquad (2).\end{aligned}\right\}$$

From (1) we get $y = x + 1$;

and, putting this value for y in (2),

we have $\frac{2x+1}{x} = \frac{3x-1}{2}$, whence $x = 2$ or $-\frac{1}{3}$,

and $\therefore y = x + 1 = 3$ or $\frac{2}{3}$.

The given equations have, therefore, two pairs of roots

$x = 2$ and $y = 3$, or $x = -\frac{1}{3}$ and $y = \frac{2}{3}$.

88. *Def.* An expression is said to be *homogeneous* with respect to x and y when the sum of the indices of x and y in each of its terms is the same; thus

$$3x^2 + 2xy - y^2,$$
$$x^3 - 2x^2y + 3xy^2,$$
$$4x^4 - 3x^3y + 6x^2y^2 - y^4, \text{ \&c.}$$

are homogeneous expressions with respect to x and y of two, three, four, &c. dimensions respectively.

Now when each of the two equations is homogeneous with respect to x and y, in all the terms which involve x and y, we may generally without difficulty obtain an equation involving only one unknown, by putting $y = vx$, where v is some quantity at present unknown.

Ex. 2. $\left.\begin{aligned} x^2+xy+y^2 &= 7 \\ 2x^2+3y^2 &= 14 \end{aligned}\right\}\quad \begin{aligned}&(1)\\&(2).\end{aligned}$

Here putting vx for y, $x^2(1+v+v^2) = 7$, (α)

$$x^2(2+3v^2) = 14; \qquad (\beta)$$

$\therefore$ dividing (α) by (β), the x^2 disappears, and we have

$$\frac{1+v+v^2}{2+3v^2} = \frac{7}{14} = \frac{1}{2}; \text{ whence } v = 2;$$

and from (β), $x^2(2+12) = 14$, or $x = \pm 1$,

$$\text{and } y = vx \qquad = \pm 2.$$

89. *Def.* An expression is said to be *symmetrical* with respect to x and y, when these quantities are similarly involved in it: thus

$$4xy + 5x + 5y - 1,$$
$$x^3 - 3x^2y^2 + y^3,$$
$$2x^4 - 3x^2y - 3xy^2 + 2y^4,$$

are symmetrical with respect to x and y.

Now, when each of the two equations is symmetrical with respect to x and y, they may often be solved, by putting $u+v$ for x and $u-v$ for y, where u and v are at present unknown.

Ex. 3. $\left.\begin{aligned} x^3+y^3 &= 18xy \\ x+y &= 12 \end{aligned}\right\}\quad \begin{aligned}&(1)\\&(2).\end{aligned}$

Put $u+v$ for x, and $u-v$ for y;

then (1) becomes $(u+v)^3 + (u-v)^3 = 18(u+v)(u-v)$,

$$\text{whence } u^3 + 3uv^2 = 9(u^2 - v^2); \quad (\alpha)$$

and (2) becomes $(u+v)+(u-v) = 12$, whence $u = 6$; putting this for u in (α), we have

$$216 + 18v^2 = 9(36 - v^2), \text{ whence } v = \pm 2;$$
$$\therefore x = u + v = 6 \pm 2 = 8 \text{ or } 4,$$
$$\text{and } y = u - v = 6 \mp 2 = 4 \text{ or } 8.$$

90. The preceding are *general* rules for the solution of equations of the kinds here referred to, and will some-

times succeed also in other equations; yet in many of these cases a little ingenuity will often suggest some step or artifice, by which the roots may be found more simply, but for which no rules can be given.

The methods followed in the two following examples are worthy of notice in this respect.

Ex. 4. $$\left.\begin{aligned}3x^2 - 2xy &= 15\\ 2x + \;\; 3y &= 12\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2).\end{aligned}$$

Mult. (1) by 3 we have $9x^2 - 6xy = 45$,

.... (2) by $2x$ ...... $4x^2 + 6xy = 24x$;

$\therefore$ adding, $13x^2 = 45 + 24x$,

or $13x^2 - 24x = 45$, whence $x = 3$ or $-1\frac{2}{13}$,

and $\therefore$ $\left(\text{since from (2) } y = \dfrac{12 - 2x}{3}\right)$ $y = 2$ or $4\frac{10}{13}$.

Ex. 5. $$\left.\begin{aligned}x^2 + y^2 &= 25\\ 2xy &= 24\end{aligned}\right\}.$$

Here adding, $x^2 + 2xy + y^2 = 49$, whence $x + y = \pm 7$,

subtracting, $x^2 - 2xy + y^2 = 1$, whence $x - y = \pm 1$;

$$\begin{array}{ll} \text{if } x + y = +7, & x + y = +7, \\ \text{and } x - y = +1, & x - y = -1; \\ \hline \therefore 2x = 8, \text{ and } x = 4, & 2x = 6, \text{ and } x = 3, \\ 2y = 6, \text{ and } y = 3, & 2y = 8, \text{ and } y = 4: \end{array}$$

similarly by combining the equation $x + y = -7$ with each of the two $x - y = \pm 1$, we should get the other two pairs of roots

$$x = -4, y = -3, \text{ and } x = -3, y = -4.$$

EXAMPLES.

390. $$\left.\begin{aligned}\frac{3x + 5y}{10} + \frac{4x - 3y}{6} &= 6\tfrac{23}{30}\\ 3x^2 + 2y^2 &= 179\end{aligned}\right\}.$$

391. $$\left.\begin{aligned}x^2 + y^2 &= 25\\ x + y &= \;\;1\end{aligned}\right\}.$$

392. $\left.\begin{aligned} x^2+y^2&=25\\ 4y+3x&=24 \end{aligned}\right\}.$

393. $\left.\begin{aligned} 2(x-y)&=11\\ xy&=20 \end{aligned}\right\}.$

394. $\left.\begin{aligned} x^2+xy&=66\\ x^2-y^2&=11 \end{aligned}\right\}.$

395. $\left.\begin{aligned} x-y&=2\\ 15(x^2-y^2)&=16xy \end{aligned}\right\}.$

396. $\left.\begin{aligned} \frac{x^2}{y^2}&=\frac{85}{9}-\frac{4x}{y}\\ x-y&=2 \end{aligned}\right\}.$

397. $\left.\begin{aligned} xy&=\left(x-\frac{3}{7}\right)\left(y+\frac{2}{3}\right)\\ x^2y^2&=(x^2+3)(y^2-4) \end{aligned}\right\}.$

398. $\left.\begin{aligned} x+y&=6\\ x^3+y^3&=72 \end{aligned}\right\}.$

399. $\left.\begin{aligned} \frac{x}{y}&=\frac{2}{3}\\ 3xy+2x+y&=485 \end{aligned}\right\}.$

400. $\left.\begin{aligned} x-y&=1\\ x^3-y^3&=19 \end{aligned}\right\}.$

401. $\left.\begin{aligned} x^3+y^3&=189\\ x^2y+xy^2&=180 \end{aligned}\right\}.$

PROBLEMS.

402. There are two numbers, one of which is $\frac{4}{5}$ of the other, and the difference of their squares is 81: find them.

403. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356: find them.

404. There are two numbers, one of which is triple of the other, and the difference of their squares is 128: find them.

405. In a certain court there are two square grass plots, a side of one of which is 10 yards longer than a side of the other, and the area of the latter is $\frac{9}{25}$ of that of the former. What are the lengths of the sides?

406. What two numbers make up 14, so that the quotient of the less divided by the greater is $\frac{9}{16}$ of the quotient of the greater divided by the less?

407. A draper bought a piece of silk for £16. 4*s.*, and the number of shillings which he paid per yard was $\frac{4}{9}$ the number of yards. How much did he buy?

408. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but on the enemy coming in sight, the front was increased by 845 men, and by this movement the detachment was drawn up in 5 lines: find the number of men.

409. What number is that, the sum of whose third and fourth parts is less by 2 than the square of its sixth part?

410. There is a number such that the product of the numbers obtained by adding 3 and 5 to it respectively is less by 1 than the square of its double: find it.

411. There is a rectangular field, whose length exceeds its breadth by 16 yards, and it contains 960 square yards: find its dimensions.

412. The difference between the hypothenuse and two sides of a right-angled triangle is 3 and 6 respectively: find the sides.

413. What two numbers are those whose difference is 5, and their sum multiplied by the greater 228?

414. A labourer dug two trenches, one 6 yards longer than the other, for £17. 16*s*., and the digging of each of them cost as many shillings *per* yard, as there were yards in its length: what was the length of each?

415. The plate of a looking-glass is 18 inches by 12, and it is to be framed with a frame of uniform width, whose area is to be equal to that of the glass: find the width of the frame.

416. There are two square buildings, paved with stones, each a foot square. The side of one building exceeds that of the other by 12 feet, and the two pave-

ments together contain 2120 stones: find the sides of the buildings.

417. A person bought a certain number of oxen for £240., and, after losing 3, sold the remainder for £8. a head more than they cost him, thus gaining £59. by the bargain: what number did he purchase?

418. A tailor bought a piece of cloth for £147., from which he cut off 12 yards for his own use, and sold the remainder for £120. 5*s.* charging 5 shillings *per* yard more than he gave for it. How many yards were there, and what did it cost him *per* yard?

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91. We have seen in (72), that when we have only *one* equation between *two* unknowns, the number of values of these by which it will be satisfied is *unlimited*, and the equation is *indeterminate*. If, however, we require only those values which are *integers*, the number of solutions will be restricted: and still more so, if we require only those which are *positive* integers.

The method of treating an indeterminate equation, under such restrictions, will appear in the following examples.

Ex. 1. Find the integral solutions of $3x + 5y = 74$.

Divide by the *lowest* coeff., and express the improper fractions which may arise as mixed numbers;

$$\text{then } x + y + \frac{2y}{3} = 24 + \frac{2}{3},$$

$$\text{or } x + y - 24 = \frac{2}{3} - \frac{2y}{3} = \frac{2 - 2y}{3} \ldots . (1).$$

Now, since $x + y - 24$ is integral, so is $\frac{2 - 2y}{3}$, and any multiple of it; multiply it then, if possible, by such a number as will make the coeff. of y divisible by the den^r with remainder 1, *i.e.* in this case, multiply by 2;

$$\text{then } \frac{4-4y}{3} \text{ or } 1-y+\frac{1-y}{3} \text{ is integral:}$$

$$\text{hence } \frac{1-y}{3} \text{ is integral} = t \text{ suppose,}$$

$$\therefore\ 1-y=3t, \text{ and } y=1-3t;$$

$$\text{and (1) } x+1-3t-24=\frac{2-2+6t}{3}=2t, \text{ or } x=23+5t:$$

from which expressions $\left.\begin{array}{l} x = 23+5t \\ y = \ \ 1-3t \end{array}\right\}$, we may get all the integral values of x and y, which satisfy the equation, by giving t *any* integral value, positive or negative.

Thus, if

$$t=-4,\quad -3,\quad -2,\quad -1,\quad 0,\quad 1,\quad 2, \text{ \&c.}$$

we have

$$\left.\begin{array}{l} x=\ \ 3 \\ y=13 \end{array}\right\} \left.\begin{array}{r} 8 \\ 10 \end{array}\right\} \left.\begin{array}{r} 13 \\ 7 \end{array}\right\} \left.\begin{array}{r} 18 \\ 4 \end{array}\right\} \left.\begin{array}{r} 23 \\ 1 \end{array}\right\} \left.\begin{array}{r} 28 \\ -2 \end{array}\right\} \left.\begin{array}{r} 33 \\ -5 \end{array}\right\}, \text{ \&c.}$$

If, however, we require only the *positive* integral values of x and y, we may not give t *any* integral value, but only those which will leave x and y *positive.*

Now, if t be taken *negative,* since $x = 23 + 5t$,

we must not have $5t > 23$, or $t > \frac{23}{5}$, that is, > 4;

and if t be taken *positive,* since $y = 1 - 3t$,

we must not have $3t > 1$, or $t > \frac{1}{3}$, that is, > 0;

hence the values for t range from -4 to 0 inclusively, and thus there will be only 5 *positive integral* solutions of the given equation, viz. the first five pairs of values, above exhibited.

Ex. 2. Find the integral solutions of $14x - 5y = 7$.

Here $2x + \frac{4x}{5} - y = 1 + \frac{2}{5}$, or $2x - y - 1 = \frac{2-4x}{5}$ ... (1);

hence $\frac{2-4x}{5}$, and $\therefore \frac{8-16x}{5}$, or $1 - 3x + \frac{3-x}{5}$ is integral;

$$\therefore \frac{3-x}{5} \text{ is integral} = t, \text{ whence } x = 3 - 5t,$$

and (1) $6 - 10t - y - 1 = \frac{2 - 12 + 20t}{5} = 4t - 2, \therefore y = 7 - 14t$:

from which expressions $\left.\begin{array}{l} x = 3 - \ \ 5t \\ y = 7 - 14t \end{array}\right\}$, we may get, as before, all the integral values of x and y.

If we take $t = 0$, we have $x = 3$, $y = 7$, which are the *least* positive integral values they admit of; but the number of *positive* values is in this case unlimited, since we may take *any* negative value for t, though we cannot take it at all positive.

Ex. 3. In how many ways may £80 be paid in sovereigns and guineas?

Let x = number of sovereigns, y = number of guineas; then $20x + 21y$ = number of shillings in £80 = 1600;

$$\therefore x + y + \frac{y}{20} = 80 \ldots\ldots (1);$$

here the coeff. of y in the num^r of $\frac{y}{20}$ being already 1,

let $\frac{y}{20} = t$; $\therefore y = 20t$, and (1) $x = 80 - 21t$.

Hence t cannot be taken *negatively* at all, nor *positively* $> \frac{80}{21}$, that is > 3: there are therefore only *four* solutions; and if $t = 0, \ 1, \ 2, \ 3,$

$$\text{we have } \begin{array}{l} x = \\ y = \end{array} \left.\begin{array}{r} 80 \\ 0 \end{array}\right\} \left.\begin{array}{r} 59 \\ 20 \end{array}\right\} \left.\begin{array}{r} 38 \\ 40 \end{array}\right\} \left.\begin{array}{r} 17 \\ 60 \end{array}\right\}.$$

EXAMPLES.

419. Find the positive integral solutions of $4x+29y=150$; of $3x+29y=151$; of $7x+15y=225$.

420. Find the least positive integral solution of $17x-7y=1$; of $23x-9y=929$; of $8x-23y=19$.

421. Find the number of positive integral solutions of $3x+4y=39$; of $8x+13y=500$; of $7x+13y=405$; of $2x+7y=125$.

422. A person distributed 4*s.* 2*d.* among some beggars, giving 7*d.* each to some, and 1*s.* each to the rest: how many were there in all?

423. In how many ways could a sum of 12 guineas be made up of half-guineas and half-crowns? In how many ways, of guineas and crowns?

424. How many fractions are there with den^{rs} 12 and 18, whose sum is $\frac{25}{36}$?

425. *A* wishes to pay *B* a debt of £1. 12*s.*, but has only half-crowns in his pocket, while *B* has only fourpenny pieces; how may they settle the matter most simply between them?

426. What is the least number, which, divided by 3 and 5, leaves remainders 2 and 3 respectively? What is the least, which, divided by 3 and 7, leaves remainders 1 and 2?

427. A person buys two pieces of cloth for £15, the one at 8*s.* the other at 11*s.* *per* yard, and each containing more than 10 yards: how many yards did he buy altogether?

428. In how many ways can £1. be paid in half-crowns, shillings, and sixpences, the number of coins used at each payment being 18?

429. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over, but when he counts them 5 at a time, there are 4 over: how many were there in all?

430. A person bought 40 animals, consisting of calves, pigs, and geese for £40; the calves cost him £5. a piece, the pigs £1, and the geese a crown: how many did he buy of each?

92. The following remarks should be attended to in the solution of these equations.

1. The equation should be first simplified, by striking out any factor common to *all* its terms; and when thus reduced to the form $ax + by = c$, (where a, b, c, represent numbers, positive or negative, as the case may be,) then, if a and b have any common factor not common also to c, there is *no* integral solution.

For let $a = md$, $b = nd$, while c does not contain d; then $mdx + ndy = c$, and $mx + ny = \frac{c}{d}$, where $\frac{c}{d}$ is a fraction: now, if x and y had any *integral* values, $mx + ny$ would be integral, and yet equal a *fraction*, which is absurd.

2. The multiplier required in the Rule will always be found to be some number *less than the den^r^*, except in the case just referred to, when no such multiplier will be found.

3. If the coeff^s^ of x and y in the given equation have the same sign, as in Ex. 1, the number of *positive* integral solutions will be limited, the expression for x determining the limit in one direction, and that for y in

the other; but if they have *different* signs, as in Ex. 2, the number of such solutions is *unlimited*, unless the equation be of the form described above, which has *no* integral solution.

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93. *Inequalities*, in which two expressions are connected by the sign > or <, are treated just like equations, with this exception, that if we change the signs of *all* the terms, (as we may do in equations,) we must also change the sign of inequality from > to <, or < to >; thus although 7 is > 3, yet -7 is < -3.

Inequalities are often reducible to the statement in (55), viz. that, whatever be the values of a and b,

$$a^2 + b^2 > 2ab.^*$$

Ex. 1. Which is greater, $m^3 + 1$ or $m^2 + m$, whatever m may be?

Here $m^3 + 1$ is $> m^2 + m$,
if $m^3 + 1 > m(m + 1)$,
if $m^2 - m + 1 > m$, (dividing each side by $m + 1$),
if $m^2 + 1 > 2m$, (transposing the $-m$),
which is the case by (55);
$\therefore m^3 + 1 > m^2 + m$.

Ex. 2. Shew that $a^3 + b^3 > a^2b + ab^2$.

$$\begin{aligned} \text{Since} \quad a^2 + b^2 > 2ab, \quad \therefore\ a^3 + ab^2 &> 2a^2b \\ \text{and} \quad a^2b + b^3 &> 2ab^2 \\ \therefore\ \text{adding,} \quad a^3 + b^3 + a^2b + ab^2 &> 2a^2b + 2ab^2 \\ \text{or} \quad a^3 + b^3 &> a^2b + ab^2. \end{aligned}$$

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[*] Strictly speaking this should be $a^2 + b^2$ not $< 2ab$; since in (55) if $a = b$, then $(a - b)^2 = 0$, and $a^2 + b^2 = 2ab$: if, however, a and b are *unequal* quantities, the statement in the text is correct. This remark may be applied to all the examples given under this Article.

EXAMPLES.

431. Shew that 5 is the integer value of x, when

$$\frac{x+2}{4} + \frac{x}{3} < \frac{x-4}{2} + 3 \text{ and } > \frac{x+1}{2} + \frac{1}{3}.$$

432. Shew that the sum of any fraction, and its reciprocal, is > unity.

433. Shew that $\frac{a+b}{2} > \frac{2ab}{a+b}$.

434. Shew that $\frac{a}{b} \gtrless \frac{a+x}{b+x}$ and $\lessgtr \frac{a-x}{b-x}$, as $a \gtrless b$.

435. Shew that $a^6 + a^4b^2 + a^2b^4 + b^6 > (a^3 + b^3)^2$.

436. Shew that $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{b} + \frac{1}{a}$.

437. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, shew that

$$xy > ac + bd \text{ or } ad + bc.$$

438. If $a > b$, shew that $a^4 - b^4 < 4a^3(a-b)$ and

$$> 4b^3(a-b).$$

439. Shew that $a^2 + b^2 + c^2 > ab + ac + bc$.

440. Shew that $a^3 + b^3 + c^3 > \frac{1}{2}\{a^2b + ab^2 + a^2c + ac^2\}$.

441. Shew that $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 > 6abc$.

442. Shew that $(a+b+c)^3 > 27abc$.

CHAPTER VII.

INDICES, SURDS, AND IMPOSSIBLE QUANTITIES.

94. It was stated in (56), that when any root of a quantity cannot be exactly obtained, it is expressed by the use of the sign of Evolution, as $\sqrt{3}$, $\sqrt[3]{2ac}$, $\sqrt[5]{a^2 + c^2}$, and called an *Irrational* or *Surd* quantity.

It was also stated in (58) that there could not be any *even* root of a *negative* quantity; but that such roots may be expressed in the form of surds, as $\sqrt{-3}$, $\sqrt[4]{-a^2}$, $\sqrt[6]{-(a^2 + b^2)}$, and are then called *impossible* or *imaginary* quantities.

These we shall consider more at length in this chapter.

95. We have seen in (20) that powers of the same quantity are multiplied by adding the indices; there, however, this rule was not proved *generally*, but only shown in particular instances.

To prove that $a^m \times a^n = a^{m+n}$, *when* m *and* n *are any positive integers.*

Since by (9) $a^m = a \times a \times \text{\&c.}$ (m factors)

and $a^n = a \times a \times \text{\&c.}$ (n factors),

it follows that

$$a^m \times a^n = a \times a \times \text{\&c.}\,(m \text{ factors}) \times a \times a \times \text{\&c.}\,(n \text{ factors})$$
$$= a \times a \times \text{\&c.}\,(m + n \text{ factors}) = a^{m+n}, \text{ by (9)}.$$

96. Hence $(a^m)^n = a^{mn} = (a^n)^m$;

for $(a^m)^n = a^m \,.\, a^m \,.\, a^m \,.\, \text{\&c. } n \text{ factors} = a^{m+m+m+\text{\&c. } n \text{ terms}} = a^{mn}$,

and $(a^n)^m = a^n \,.\, a^n \,.\, a^n \,.\, \text{\&c. } m \text{ factors} = a^{n+n+n+\text{\&c. } m \text{ terms}} = a^{nm}$;

$\therefore$ since $a^{mn} = a^{nm}$, we have $(a^m)^n = a^{mn} = (a^n)^m$;

that is, *the* nth *power of the* mth *power of* a = *the* mth *power of the* nth *power of* a, and either is found by multiplying the two indices.

97. Hence also $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$;

for let $\sqrt[n]{a^m} = x^m$, then $a^m = (x^m)^n = (x^n)^m$ by (96);

hence $a = x^n$, and $\therefore \sqrt[n]{a} = x$ and $(\sqrt[n]{a})^m = x^m$;

$\therefore$ since $x^m = \sqrt[n]{a^m}$, we have $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$:

that is, *the n*th *root of the m*th *power of* a = *the* mth *power of the* nth *root of* a.

98. These results have reference as yet only to positive integral indices, which (9) were first used to express briefly the repetition of the same factor in any product.

In time, however, this question arose: If we write down a quantity, with a positive *fraction* for an index, such as $a^{\frac{p}{q}}$, and agree that such a symbol shall be treated by the same law of Multiplication as if the index were an *integer*, viz. $a^m . a^n = a^{m+n}$ (95)—what would such a symbol, so treated, denote?

The answer is thus given: Since (96) it follows immediately from this law, in the case of positive *integers*, that $(a^m)^n = a^{mn}$, we have here also, according to our supposition,

$$(a^{\frac{p}{q}})^q = a^{\frac{pq}{q}} = a^p;$$

and hence it appears, that $a^{\frac{p}{q}}$ denotes such a quantity as, *when raised to the* qth *power,* becomes equal to a^p. But that quantity, whose qth power = a^p, is (10) *the* qth *root of* a^p; and, therefore, $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$ by (97).

Hence, when a fractional index is employed with any quantity, the *num^r* denotes a *power*, and the *den^r* a *root* to be taken of it: thus,

$a^{\frac{1}{2}} = 2^{\text{nd}}$ root of 1^{st} power of $a = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, &c.

$$\text{and } a^{\frac{2}{3}} = 3^{\text{rd}} \text{ root of } 2^{\text{nd}} \text{ power of } a = \sqrt[3]{a^2},$$

$$\text{or } = 2^{\text{nd}} \text{ power of } 3^{\text{rd}} \text{ root of } a = (\sqrt[3]{a})^2;$$

$$a^{\frac{3}{4}} = \sqrt[4]{a^3} \text{ or } (\sqrt[4]{a})^3, \quad a^{\frac{2}{5}} = \sqrt[5]{a^2} \text{ or } (\sqrt[5]{a})^2, \text{ \&c.}$$

We may notice also that $a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} = \text{\&c.}$,

$$\text{or } \sqrt{a} = \sqrt[4]{a^2} = \sqrt[6]{a^3} = \text{\&c.}$$

99. If we now write down a quantity with a *negative* index, such as a^{-p}, (where p may be integral or fractional) and agree that this symbol shall be treated by the same law of Multiplication as if the index were positive—what would such a symbol, so treated, denote?

The answer is thus given:

Since this law is $a^m . a^n . = a^{m+n}$, we have

$$a^{m+p} . a^{-p} = a^{m+p-p} = a^m:$$

$$\text{but } a^{m+p} \div a^p = \frac{a^{m+p}}{a^p} = \frac{a^m . a^p}{a^p} = a^m;$$

hence, to *multiply* a quantity by a^{-p}, is the same as to *divide* it by a^p; so that

$$1 \times a^{-p} = 1 \div a^p, \quad \text{or} \quad a^{-p} = \frac{1}{a^p}.$$

Hence, any quantity with a *negative* index denotes the *reciprocal* of the same quantity with the same *positive* index: thus,

$$a^{-1} = \frac{1}{a}, \quad a^{-2} = \frac{1}{a^2}, \quad a^{-3} = \frac{1}{a^3}, \text{ \&c.}$$

$$\text{and } a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}, \qquad a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}},$$

$$\text{or } = \sqrt{a^{-1}}, \qquad \text{or } = \sqrt[3]{a^{-2}}.$$

Hence also any power in the num^r of a quantity may be removed into the den^r, and *vice versâ*, by changing the sign of its index; thus

$$a^{-3}b^2c^{-1} = \frac{a^{-3}b^2}{c} = \frac{a^{-3}}{b^{-2}c} = \frac{b^2c^{-1}}{a^3} = \&c.$$

$$\frac{a^{-\frac{1}{2}}b^{-\frac{2}{3}}}{c^{-\frac{3}{4}}} = \frac{c^{\frac{3}{4}}}{a^{\frac{1}{2}}b^{\frac{2}{3}}} = \frac{1}{a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-\frac{3}{4}}} = a^{-\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{3}{4}} = \&c.$$

100. Lastly, if we write down a quantity with *zero* for an index, as a^0, and agree that this symbol shall be treated as if the index were an actual number—what then would it denote?

Since $a^m \times a^{-m} = a^{m-m} = a^0$, and also $a^m \times \frac{1}{a^m} = \frac{a^m}{a^m} = 1$, it follows that a^0 will denote the number 1, whatever be the value of a.

EXAMPLES.

Express, with *fractional* indices,

443. $\sqrt{x^3} + 2\sqrt[3]{x^4} + 3(\sqrt{x})^5 + 4(\sqrt[3]{x})^2$.

444. $\sqrt[3]{a^4} + 4\sqrt[3]{a^3b^2} + 6\sqrt[3]{a^2b^4} + 4\sqrt[3]{ab^6} + a\sqrt[3]{b^8}$.

445. $(\sqrt{a})^5 + 2\sqrt[3]{a^6b} + 4\sqrt[6]{a^6b^4} + 8\sqrt[4]{a^2b^8} + 5a(\sqrt[3]{b})^6$.

Express, with negative indices, so as to remove all powers, (1) into the num^{rs}, and (2) into the den^{rs},

446. $\frac{1}{a} + \frac{2}{b^2} + \frac{3}{c^3}$; $\frac{a}{b} + \frac{b}{a}$; $\frac{a^3}{b^3} + \frac{3a^2}{b} + \frac{4b}{a^2} + \frac{2b^2}{a^3}$.

447. $\frac{a^3}{3b^2c^2} + \frac{4c^2}{a^2b} + \frac{2bc}{a} + \frac{1}{3abc}$.

448. $\frac{ab}{2\sqrt[3]{c}} + \frac{2b^2c^2}{3\sqrt{a^3}} + \frac{3}{4\sqrt[3]{a^2bc^3}} + \frac{5c}{a\sqrt[4]{b^3}}$.

Express, with the *sign of Evolution*,

449. $a^{\frac{1}{2}} + 2a^{\frac{2}{3}} + 3a^{\frac{3}{4}} + 4a^{\frac{1}{5}}$. 450. $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{3}{4}} + b^{\frac{3}{2}}$.

451. $\dfrac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{2c^{\frac{1}{2}}} + \dfrac{2a^{\frac{1}{4}}c^{\frac{3}{4}}}{3b^{\frac{3}{2}}} + \dfrac{a^{\frac{1}{5}}b^{\frac{2}{5}}c^{\frac{3}{5}}}{4} + \dfrac{b^{\frac{1}{6}}c^{\frac{5}{6}}}{5a^{\frac{3}{4}}}$.

Express, with *positive* indices, and, where fractional, with the sign of Evolution,

452. $a^{-1}bc + 2ab^{-2}c + 3a^{-1}b^{-1}c^{-1} + 4a^{-1}b^{-2}c^{3}$.

453. $\dfrac{a^{-2}b^{-2}}{c^{-1}} + \dfrac{2a}{b^{-1}c^{-1}} + \dfrac{3b^{-1}c^{-2}}{a^{-3}} + \dfrac{1}{a^{-1}b^{-2}c^{-3}}$.

454. $a^{-\frac{2}{5}} + \dfrac{a^{-2}}{b^{-\frac{1}{3}}} + \dfrac{b^{-\frac{2}{3}}}{a^{-\frac{3}{2}}} + \dfrac{b}{a^{-1}} + a^{\frac{1}{2}}b^{-\frac{4}{3}} + b^{-\frac{5}{3}}$.

101. From the preceding it appears that, *whatever* be the indices,

$$a^m \times a^n = a^{m+n},$$

and

$$\frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m \times a^{-n} = a^{m-n};$$

so that to *multiply* any powers of the same quantity, we must *add* the indices; and to *divide* any one power of a quantity by another, we must *subtract* the index of the divisor from that of the dividend.

Thus $a^3 \times a^{-2} = a^{3-2} = a$, $a^3 \div a^{-\frac{1}{2}} = a^{3+\frac{1}{2}} = a^{\frac{7}{2}}$,

$$a^{-\frac{1}{2}} \div a^{-\frac{3}{5}} = a^{-\frac{1}{2}+\frac{3}{5}} = a^{\frac{-5+6}{10}} = a^{\frac{1}{10}}.$$

Ex. 1. *Multiplication.*

$$\begin{array}{l}
a^{\frac{5}{2}} + a^{2}b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} \\
a^{\frac{1}{2}} - b^{\frac{1}{3}} \\
\hline
a^{3} + a^{\frac{5}{2}}b^{\frac{1}{3}} + a^{2}b^{\frac{2}{3}} + a^{\frac{3}{2}}b + ab^{\frac{4}{3}} + a^{\frac{1}{2}}b^{\frac{5}{3}} \\
\qquad - a^{\frac{5}{2}}b^{\frac{1}{3}} - a^{2}b^{\frac{2}{3}} - a^{\frac{3}{2}}b - ab^{\frac{4}{3}} - a^{\frac{1}{2}}b^{\frac{5}{3}} - b^{2} \\
\hline
a^{3} \quad * \quad * \quad * \quad * \quad * \quad - b^{2}
\end{array}$$

Ex. 2. *Division.*

$$\begin{array}{l}
x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2)\,x^{\frac{5}{2}} - x^2 - 4x^{\frac{3}{2}} + 6x - 2x^{\frac{1}{2}}\,(x - x^{\frac{1}{2}} \\
\qquad\qquad\quad \underline{x^{\frac{5}{2}} \qquad\; - 4x^{\frac{3}{2}} + 2x} \\
\qquad\qquad\qquad - x^2 \qquad + 4x - 2x^{\frac{1}{2}} \\
\qquad\qquad\qquad - x^2 \qquad + 4x - 2x^{\frac{1}{2}}
\end{array}$$

EXAMPLES.

455. Multiply $x + 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$ by $x - 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$.

456. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

457. Multiply $a^{\frac{5}{2}} - 2a^2b^{\frac{1}{3}} + 4a^{\frac{3}{2}}b^{\frac{2}{3}} - 8ab + 16a^{\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$
by $a^{\frac{1}{2}} + 2b^{\frac{1}{3}}$.

458. Multiply $a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$ by $a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$.

459. Multiply $x^{\frac{5}{8}} + x^{\frac{1}{2}}y^{-\frac{1}{8}} + x^{\frac{3}{8}}y^{-\frac{1}{4}} + x^{\frac{1}{4}}y^{-\frac{3}{8}} + x^{\frac{1}{8}}y^{-\frac{1}{2}} + y^{-\frac{5}{8}}$
by $x^{\frac{3}{8}} - x^{\frac{1}{4}}y^{-\frac{1}{8}} + x^{\frac{1}{8}}y^{-\frac{1}{4}} - y^{-\frac{3}{8}}$.

460. Divide $16x - y^2$ by $2x^{\frac{1}{4}} - y^{\frac{1}{2}}$.

461. Divide $x^{-1} - y^{-1}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.

462. Divide $a^{-3} - 64b^2$ by $a^{-\frac{1}{2}} + 2b^{\frac{1}{3}}$.

463. Divide $x - 2x^{\frac{1}{2}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$.

464. Divide $8a^{\frac{3}{2}} + b^{-\frac{3}{2}} - c + 6a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{3}}$
by $2a^{\frac{1}{2}} + b^{-\frac{1}{2}} - c^{\frac{1}{3}}$.

465. Find the cubes of

$$a^{\frac{1}{3}}b^{-1} + a^{-\frac{1}{3}}b \text{ and } \frac{2x^{\frac{2}{3}}y^{-\frac{1}{3}}}{3} - \frac{3x^{-\frac{1}{3}}y^{\frac{2}{3}}}{2}.$$

466. Find the cube of $a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{6}} + 3b^{\frac{1}{3}}$.

467. Find the fourth power of $x^{\frac{3}{4}} - y^{\frac{5}{2}}$.

468. Find the fourth power of $a^{\frac{1}{2}}b^{-\frac{1}{2}} - a^{-\frac{1}{2}}b^{\frac{1}{2}}$.

469. Find the square root of

$$a^2b^{-2} + 2ab^{-1} + 3 + 2a^{-1}b + a^{-2}b^2.$$

470. Find the square root of

$$a^{\frac{4}{3}} - 3a + \frac{33a^{\frac{2}{3}}}{4} - 21a^{\frac{1}{3}} + 45 - 63a^{-\frac{1}{3}}$$

$$+ 90a^{-\frac{2}{3}} - 108a^{-1} + 81a^{-\frac{4}{3}}.$$

471. Find the fourth root of

$$x^4y^{-\frac{4}{3}} - 4x^{\frac{5}{2}}y^{-\frac{1}{3}} + 6xy^{\frac{2}{3}} - 4x^{-\frac{1}{2}}y^{\frac{5}{3}} + x^{-2}y^{\frac{8}{3}}.$$

472. Find the fourth root of

$$16x^6 - 96x^{\frac{9}{2}}y^{\frac{3}{4}} + 216x^3y^{\frac{3}{2}} - 216x^{\frac{3}{2}}y^{\frac{9}{4}} + 81y^3.$$

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102. Since every fractional index indicates by its den^r a root to be extracted, all quantities having such indices are expressed as surds.

When a *negative* quantity has the den^r of its index (reduced to its lowest terms) *even* (58), the expression will be imaginary; thus $\sqrt{-3}$ or $(-3)^{\frac{1}{2}}$, $\sqrt[4]{-9}$ or $(-9)^{\frac{1}{4}}$, are imaginary quantities; but $(-4)^{\frac{4}{6}}$ is not so, since it $= (-4)^{\frac{2}{3}}$, where the root to be taken is odd.

103. In the case of a *numerical* surd, expressed with a fractional index, should the num^r be any other than *unity*, we may take at once the required power, and so have unity only for the num^r, and a simple root to be extracted; thus

$$2^{\frac{2}{3}} = (2^2)^{\frac{1}{3}} = 4^{\frac{1}{3}} \text{ or } \sqrt[3]{4},\ 3^{-\frac{3}{5}} = (3^{-3})^{\frac{1}{5}} = \left(\frac{1}{27}\right)^{\frac{1}{5}} \text{ or } \sqrt[5]{\frac{1}{27}}\,.$$

104. Quantities are often expressed in the *form* of surds, which are not really so, *i. e.* when we *can* extract the roots indicated. Thus $\sqrt{a}$, $\sqrt[3]{7}$, $(a^2 + ab + b^2)^{\frac{1}{2}}$ are *actually* surds, whose roots we cannot obtain; but $\sqrt{a^2}$, $\sqrt[3]{27}$, $(4a^2 + 4ab + b^2)^{\frac{1}{2}}$ are only *apparently* so, and are respectively $= a,\ 3,\ 2a + b$.

Conversely, any rational quantity may be expressed in the form of a surd, by raising it to the power indicated by the den^r of the surd-index. Thus

$$2 = 4^{\frac{1}{2}} = 8^{\frac{1}{3}} = \sqrt[4]{16} = \&c.$$

$$a = \sqrt[3]{a^3},\ \frac{3c}{2} = \left(\frac{9c^2}{4}\right)^{\frac{1}{2}},\ a + x = (a^2 + 2ax + x^2)^{\frac{1}{2}}.$$

105. In like manner, a *mixed* surd, *i.e.* a product, partly rational and partly surd, may be expressed as an entire surd, by raising the rational factor to the power indicated by the den^r of the surd-index, and placing beneath the sign of Evolution the product of this power and the surd-factor. Thus

$$2\sqrt{3} = \sqrt{4} \times \sqrt{3} = \sqrt{12},\ 3.2^{\frac{2}{3}} = 3\sqrt[3]{4} = \sqrt[3]{108},$$

$$2a\sqrt{b} = \sqrt{4a^2b},\ 4a\sqrt[3]{\frac{c}{2a}} = \sqrt[3]{\frac{64a^3c}{2a}} = \sqrt[3]{32a^2c}.$$

Conversely, a surd may often be reduced to a mixed form, by separating the quantity beneath the sign of Evolution into factors, of one of which the root required may be obtained, and set outside the sign. Thus

$$\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5},\ \sqrt[3]{24} = \sqrt[3]{8 \times 3} = 2\sqrt[3]{3},$$

$$\sqrt{\frac{48a^3b}{25}} = \frac{4a}{5}\sqrt{3ab},\quad \sqrt[4]{\frac{32a^5b^4c^3}{81}} = \frac{2ab}{3}\sqrt[4]{2ac^3}.$$

106. A surd is reduced to its simplest form, when the quantity beneath the root, or *surd-factor*, is made as *small* as possible, but still *integral.*

Compound surd quantities are multiplied according to the method of rational quantities.

Ex. 1.
$$\begin{array}{l} 2+\sqrt{3} \\ 2-\sqrt{3} \\ \hline 4+2\sqrt{3} \\ \quad -2\sqrt{3}-3 \\ \hline 4 \quad * \quad -3=1. \end{array}$$

Ex. 2.
$$\begin{array}{l} 2+\sqrt{3} \\ 3-\sqrt{2} \\ \hline 6+3\sqrt{3} \\ \quad -2\sqrt{2}-\sqrt{6} \\ \hline 6+3\sqrt{3}-2\sqrt{2}-\sqrt{6}. \end{array}$$

Ex. 3. Find the fourth power of $\sqrt{2}+\sqrt{5}$.

$$(\sqrt{2}+\sqrt{5})^2 = 2+2\sqrt{10}+5 = 7+2\sqrt{10},$$

$$\text{and } (7+2\sqrt{10})^2 = 49+28\sqrt{10}+40 = 89+28\sqrt{10}.$$

110. *Division* of surds is performed when the divisor is a simple quantity, by a process similar to that for multiplication: thus,

$$\frac{8\sqrt{2}-12\sqrt{3}+3\sqrt{6}-4}{2\sqrt{6}} = 4\sqrt{\frac{2}{6}}-6\sqrt{\frac{3}{6}}+\frac{3}{2}-\frac{2}{\sqrt{6}}$$

$$= \tfrac{4}{3}\sqrt{3}-3\sqrt{2}+\tfrac{3}{2}-\tfrac{1}{3}\sqrt{6} \text{ (106)},$$

$$\frac{2\sqrt{3}-6\sqrt[3]{2}}{\sqrt{6}} = 2\sqrt{\frac{3}{6}}-6\sqrt[6]{\frac{4}{216}} = \sqrt{2}-\sqrt[6]{864}.$$

111. If, however, the divisor be *compound*, the division is not so easily performed; but as the form, in which compound surds usually occur, is that of a binomial quadratic surd, (*i. e.* a binomial, one or both of whose terms are surds, in which the *square* root is to be taken, such as $3+2\sqrt{5}$, $2\sqrt{3}-3\sqrt{5}$,) we shall here explain how, in all such cases, the operation of division may be converted into one of multiplication.

Since $(a+b)(a-b)=a^2-b^2$, it follows that if both num^r and den^r of any fraction, whose den^r is originally $a+b$ or $a-b$, be multiplied by $a-b$ or $a+b$ respectively, the result will be an equivalent fraction, whose den^r is

$a^2 - b^2$. Now if a and b represent *quadratic* surds, a^2 and b^2 will be rational, and therefore $a^2 - b^2$ will be rational; and we shall only have to perform the multiplication required in the num^r. Thus

$$\frac{2+\sqrt{3}}{3+\sqrt{3}} = \frac{(2+\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{6+3\sqrt{3}-2\sqrt{3}-3}{9-3} = \frac{3+\sqrt{3}}{6}.$$

$$\frac{1}{2\sqrt{2}-\sqrt{3}} = \frac{2\sqrt{2}+\sqrt{3}}{8-3} = \frac{2\sqrt{2}+\sqrt{3}}{5}.$$

Fractions of this kind are reduced to their simplest form, when the surds have been thus removed from the den^r (106).

EXAMPLES.

483. Compare the values of $6\sqrt{3}$ and $4\sqrt{7}$; $3\sqrt[3]{3}$ and $2\sqrt[3]{10}$; $2\sqrt[3]{15}$, $4\sqrt[3]{2}$, and $3\sqrt[3]{5}$.

484. Compare the values of $\sqrt{5}$ and $\sqrt[3]{11}$; $\frac{1}{2}\sqrt{2}$ and $\frac{1}{3}\sqrt{27}$; $\sqrt{5}$, $2\sqrt[3]{\frac{3}{2}}$ and $3(4\frac{1}{2})^{-\frac{1}{6}}$.

485. Find the value of $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$.

486. Find the value of $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}}$.

487. Find the value of $\sqrt[3]{72} - 3\sqrt[3]{\frac{1}{3}} + 6\sqrt[3]{21\frac{1}{3}}$.

488. Find the value of $\sqrt[3]{40} - \frac{1}{2}\sqrt[3]{320} + \sqrt[3]{135}$.

489. Find the product of $3\sqrt{8}$ and $2\sqrt{6}$; of $3\sqrt{15}$ and $4\sqrt{20}$; of $2\sqrt[3]{4}$ and $3\sqrt[3]{54}$.

490. Find the continued product of $3\sqrt{8}$, $2\sqrt[3]{6}$, and $3\sqrt[4]{54}$; of $2\sqrt{24}$, $3\sqrt[4]{18}$, and $4\sqrt[6]{24}$.

491. Multiply $3\sqrt{3} + 2\sqrt{2}$ by $\sqrt{3} - \sqrt{2}$; and $2\sqrt{15} - \sqrt{6}$ by $\sqrt{5} + 2\sqrt{2}$.

492. Find the continued product of $4 + 2\sqrt{2}$, $1 - \sqrt{3}$, $4 - 2\sqrt{2}$, $\sqrt{2} + \sqrt{3}$, $1 + \sqrt{3}$, and $\sqrt{2} - \sqrt{3}$.

493. Divide $2\sqrt{3}+3\sqrt{2}+\sqrt{30}$ by $3\sqrt{6}$, and $2\sqrt{3}+3\sqrt[3]{2}+\sqrt[4]{30}$ by $3\sqrt{2}$.

494. Reduce $\frac{1}{2\sqrt{2}-\sqrt{3}}$, $\frac{4}{\sqrt{5}-1}$, and $\frac{3}{\sqrt{5}+\sqrt{2}}$ to their simplest forms.

495. Divide $2+4\sqrt{7}$ by $2\sqrt{7}-1$; $3+2\sqrt{5}$ by $2\sqrt{5}-1$; and $5-2\sqrt{6}$ by $6-2\sqrt{6}$.

496. Reduce $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$, $\frac{3+\sqrt{5}}{3-\sqrt{5}}$, and $\frac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$ to their simplest forms.

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112. There is no general method for extracting the square or other root of *any* surd quantity; since by the coalescing of many terms into one, as in 109, Ex. 3, any power of a binomial surd will be also expressed as a binomial surd, in which all traces of the *several* terms, which formed it, will have disappeared. But we can find, as follows, the square root of a binomial surd, *one of whose terms is rational, and the other a quadratic surd;* premising, first, the following propositions, in which we use a, b, to represent rational quantities, and $\sqrt{a}$, $\sqrt{b}$, $\sqrt{x}$, $\sqrt{y}$ to represent quadratic surds.

(1) *The product of two dissimilar surds cannot be rational.*

Let $\sqrt{x}\times\sqrt{y}=m$, a rational quantity; $\therefore\ xy=m^2$;

$$\text{hence } y=\frac{m^2}{x}=\frac{m^2}{x^2}\,x, \text{ and } \sqrt{y}=\frac{m}{x}\sqrt{x},$$

or $\sqrt{y}$ may be made to have the same surd-factor as $\sqrt{x}$; that is, if $\sqrt{x}\times\sqrt{y}$ is rational, $\sqrt{x}$ and $\sqrt{y}$ must be *similar* surds (106).

(2) *A surd cannot = the sum or difference of a rational quantity and a surd, or that of two dissimilar surds.*

For let $\sqrt{a} = x \pm \sqrt{y}$, $\therefore\ a = x \pm 2x\sqrt{y} + y$;
whence $\pm 2x\sqrt{y} = a - x - y$, and $\pm\sqrt{y} = \dfrac{a - x - y}{2x}$;
or a surd = a rational quantity, which is absurd.

Again, let $\sqrt{a} = \sqrt{x} \pm \sqrt{y}$, $\therefore\ a = x \pm 2\sqrt{xy} + y$,
whence $\pm 2\sqrt{xy} = a - x - y$, and $\pm\sqrt{xy} = \dfrac{a - x - y}{2}$,
or the product of two dissimilar surds = a rational quantity, which is impossible.

(3) *If* $\mathrm{a} + \sqrt{\mathrm{b}} = \mathrm{x} + \sqrt{\mathrm{y}}$, *then* $\mathrm{a} = \mathrm{x}$, *and* $\sqrt{\mathrm{b}} = \sqrt{\mathrm{y}}$.

If not, let $a = x \pm \alpha$;

then $x \pm \alpha + \sqrt{b} = x + \sqrt{y}$, $\therefore\ \sqrt{y} = \pm\alpha + \sqrt{b}$;

or a surd = the sum or difference of a rational quantity and a surd, which is impossible; hence $a = x$, and $\therefore$ $\sqrt{b} = \sqrt{y}$.

COR. Hence if $a+\sqrt{b}=x+\sqrt{y}$, we have also $a-\sqrt{b}=x-\sqrt{y}$.

113. *If* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

For since $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, we have, squaring,
$a + \sqrt{b} = x + 2\sqrt{xy} + y$; $\therefore\ a = x + y$, and $\sqrt{b} = 2\sqrt{xy}$;
whence $a - \sqrt{b} = x - 2\sqrt{xy} + y$, and $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.
So, if $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, we shall have $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$: or the preceding statement is true, whether $\sqrt{x}$ be an *actual* or only an *apparent* surd. (104)

114. *To extract the square root of a binomial surd, one of whose terms is rational, and the other a quadratic surd.*

Let $a + \sqrt{b}$ represent the given surd;
assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, $\therefore\ \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$; (113),

and, multiplying, $\sqrt{a^2 - b} = x - y$:

but, as in (113) $a = x + y$;

$$\text{hence } a + \sqrt{a^2 - b} = 2x, \text{ and } x = \frac{a + \sqrt{a^2 - b}}{2},$$

$$a - \sqrt{a^2 - b} = 2y, \text{ and } y = \frac{a - \sqrt{a^2 - b}}{2},$$

$$\text{and } \sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}};$$

which results are true (113) whether $\sqrt{x}$ be an *actual* or *apparent* surd, *i. e.* whether the root required be the sum of two surds, or of a rational quantity and a surd.

Ex. Find the square root of $7 \pm 2\sqrt{10}$.

Let $\sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}$, $\therefore \sqrt{7 - 2\sqrt{10}} = \sqrt{x} - \sqrt{y}$;

and $\sqrt{49 - 40} = x - y$, whence $3 = x - y$

and $7 = x + y$ (113),

$\therefore 10 = 2x$ and $x = 5$, $4 = 2y$ and $y = 2$;

and $\sqrt{7 \pm 2\sqrt{10}} = \sqrt{5} \pm \sqrt{2}$.

N.B. Unless a and b are such that $a^2 - b$ is a perfect square, and so $\sqrt{a^2 - b}$ rational, the result of (114) will be useless; since the values obtained for $\sqrt{x}$ and $\sqrt{y}$, being each a *complex* surd (*i. e.* a surd within a surd), the expression $\sqrt{x} \pm \sqrt{y}$ will be more complicated than the original *single* complex surd, $\sqrt{a \pm \sqrt{b}}$. But if $a^2 - b = c^2$, a perfect square, we have $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + c}{2}} \pm \sqrt{\frac{a - c}{2}}$, or the given *complex* surd = sum of two *simple* ones.

EXAMPLES.

Find the square roots of

497. $4 + 2\sqrt{3}$.

498. $11 + 6\sqrt{2}$.

499. $6 - 2\sqrt{5}$.

500. $8 - 2\sqrt{15}$.

501. $38 - 12\sqrt{10}$. 502. $41 - 24\sqrt{2}$.

503. $2\frac{1}{4} - \sqrt{5}$. 504. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$.

505. $\frac{137}{144} - \frac{1}{3}\sqrt{2}$. 506. $13\frac{1}{8} - 3\sqrt{6}$.

Find the fourth roots of

507. $17 + 12\sqrt{2}$. 508. $193 - 132\sqrt{2}$.

509. $217 - 88\sqrt{6}$. 510. $48\frac{1}{16} + \frac{23}{2}\sqrt{15}$.

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115. Quadratic equations may often have surd roots.

Ex. $6x^2 - 44x + 33 = 0$, or $x^2 - \dfrac{22x}{3} + \dfrac{11}{2} = 0$.

$$\text{Here } x^2 - \frac{22x}{3} + \left(\frac{11}{3}\right)^2 = \frac{121}{9} - \frac{11}{2} = \frac{242 - 99}{18} = \frac{143^*}{18} = \frac{286}{36};$$

$$\text{and } x = \frac{11}{3} \pm \frac{\sqrt{286}}{6} = \frac{1}{6}.(22 \pm \sqrt{286});$$

where sum of roots $= \frac{1}{6}(22 + \sqrt{286}) + \frac{1}{6}(22 - \sqrt{286}) = \frac{44}{6} = \frac{22}{3}$, and product of roots $= \frac{1}{36}(484 - 286) = \frac{198}{36} = \frac{11}{2}$. (84)

116. As in (82) an equation may be of the quadratic form, (*i. e.* with the index of x twice as great in one term as in the other,) when the indices are negative or fractional, and may be treated accordingly.

Ex. $x^{-1} + x^{-\frac{1}{2}} = 2$.

Here $x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$, $\therefore x^{-\frac{1}{2}} = -\frac{1}{2} \pm \frac{3}{2} = 1$ or -2;

whence $x^{\frac{1}{2}} = 1$ or $-\frac{1}{2}$, and $x = 1$ or $\frac{1}{4}$.

117. If the unknown occur in an equation in the form of a *compound* surd, as in $\sqrt{12 + x} = 2 + \sqrt{x}$, we cannot immediately separate the powers of x, as required for solution of the equation. In such a case transpose all

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* When, as here, the common den^r is not a *square* number, we may make it such by multiplying both num^r and den^r by the same number; thus $\frac{143}{18} = \frac{286}{36}$.

the *rational* quantities and *simple* surds to one side; then taking powers of both sides, as the case may require, we shall probably find terms disappearing, so that we can now solve the equation, or reduce it still further, by a similar process.

Ex. $x = \sqrt{(16 + x\sqrt{x^2 + 9})} - 4.$

Here $x+4=\sqrt{(16+x\sqrt{x^2+9})}$, whence (squaring) we have $x^2+8x+16=16+x\sqrt{x^2+9}$, and $\therefore\ x^2+8x=x\sqrt{x^2+9}$; $\div$ by x {whence $x=0$ is a root (86)} $x+8=\sqrt{x^2+9}$;

$$\therefore\ x^2+16x+64=x^2+9, \text{ and } x=-\tfrac{55}{16}=-3\tfrac{7}{16}.$$

EXAMPLES.

511. $3x^2 - 2x = 2.$

512. $x + \dfrac{10}{3} = \dfrac{x^2}{12}.$

513. $\dfrac{1}{3} + \dfrac{1}{3+x} + \dfrac{1}{3+2x} = 0.$

514. $\dfrac{12}{5-x} + \dfrac{4}{4-x} = \dfrac{32}{x+2}.$

515. $x^{-2} - 2x^{-1} = 8.$

516. $2\sqrt{x} + \dfrac{2}{\sqrt{x}} = 5.$

517. $x^{-\frac{3}{4}} + \dfrac{27}{x^{-\frac{3}{4}}} = 28.$

518. $x^{-\frac{1}{3}} + 2 = \dfrac{x^{-1}+8}{x^{-\frac{2}{3}}+5}.$

519. $\dfrac{x^{-1}}{1+x^{-1}} + \dfrac{1-x^{-1}}{x^{-1}} = \dfrac{1}{6}.$

520. $4x^{\frac{2}{3}} = \dfrac{9x^{\frac{2}{3}}-4}{x^{\frac{2}{3}}-2}.$

521. $5\left(x^{-\frac{1}{4}} - 1\right)^2 = 2\left(x^{-\frac{1}{2}} + 1\right).$

522. $\sqrt{12+x} = 2 + \sqrt{x}.$

523. $\sqrt{x} + \sqrt{x+3} = \dfrac{12}{\sqrt{x+3}}.$

524. $x^2 = 21 + \sqrt{x^2 - 9}.$

525. $x + 4 + \sqrt{\dfrac{x+4}{x-4}} = \dfrac{12}{x-4}.$

526. $\sqrt{\{x^2 + \frac{1}{2}\sqrt{x^2 + 1664}\}} = x + 1.$

527. $2\sqrt{x} + \sqrt{\{4x + \sqrt{7x + 2}\}} = 1.$

528. $\sqrt{(3 + \sqrt{x})} + \sqrt{(4 - \sqrt{x})} = \sqrt{(7 + 2\sqrt{x})}.$

529. $\sqrt{3 + x} + \sqrt{3 - x} = \dfrac{36}{5\sqrt{3 + x}}.$

530. $1 + \sqrt[3]{(2x + 7)(4\sqrt{x} - 7)} = 2\sqrt{x}.$

118. *Imaginary* quantities are treated like ordinary surds; but we must here make one observation with respect to them of some importance. Since the square root of a quantity is that quantity which, *being squared, will produce the given one,* it follows that the square of $\sqrt{-a}$ will produce $-a$, i. e. $\sqrt{-a} \times \sqrt{-a} = -a$. But now by mult. of surds, we have

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{+a^2} = \pm a;$$

and how is this difference to be explained?

The fact is, that when we write $\sqrt{a^2} = \pm a$, it is not always meant that $\sqrt{a^2}$ = *both* $+a$ and $-a$, but sometimes that it equals *one* of the two, but *which* we cannot say, for want of knowing how the quantity a^2 arose, viz. whether by the product of $+a$ and $+a$ or of $-a$ and $-a$; since, if it had been the former, then $\sqrt{a^2}$ would simply $= +a$; if the latter, then $\sqrt{a^2}$ would $= -a$.

Sometimes, as in the solution of equations, there is no reason why we should take one sign only of the root, and not the other, and then we shall use them both, producing two distinct values of the root; sometimes there may be some such reason assignable, but unknown to us at present, and then we shall prefix the double sign, in uncertainty as to which of the two really belongs to it: but, as aforesaid, whenever we *do* know the manner in which a power, whose root is to be taken, has actually arisen, we may at once prefix to the root its proper sign.

Since, then, we know that the product of $\sqrt{-a} \times \sqrt{-a}$, or $\sqrt{a^2}$, denotes here $\sqrt{(-a)^2}$, we must not take the value, $+a$, of $\sqrt{a^2}$ in this case, but the value, $-a$.

119. So also $\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a) \times (-b)}$, or $\sqrt{ab}$, but this may be $+\sqrt{ab}$ or $-\sqrt{ab}$; writing, however, $\sqrt{-a}$ and $\sqrt{-b}$ in the forms $\sqrt{a} \times \sqrt{-1}$ and $\sqrt{b} \times \sqrt{-1}$, we have their product $\sqrt{ab} \times (\sqrt{-1})^2$, or $-\sqrt{ab}$, which determines, therefore, the sign of $\sqrt{-a} \times \sqrt{-b}$.

In practice it is often best to express imaginary quantities in the form just employed for $\sqrt{-a}$ and $\sqrt{-b}$, viz. by means of the factor $\sqrt{-1}$; and it should be noticed that

$$\sqrt{-1} = \sqrt{-1},\ (\sqrt{-1})^2 = -1,\ (\sqrt{-1})^3 = -\sqrt{-1},\ (\sqrt{-1})^4 = +1,$$

and so on for higher powers.

Ex. 1. $2\sqrt{-3} \times 3\sqrt{-2} = 2\sqrt{3}\sqrt{-1} \times 3\sqrt{2}\sqrt{-1} = -6\sqrt{6}$,

$$\frac{6\sqrt{-3}}{2\sqrt{-4}} = \frac{6\sqrt{3}\sqrt{-1}}{2\sqrt{4}\sqrt{-1}} = 3\sqrt{\frac{3}{4}} = \frac{3}{2}\sqrt{3}.$$

Ex. 2. Divide $4 + \sqrt{-2}$ by $2 - \sqrt{-2}$.

Here (111) $$\frac{4+\sqrt{-2}}{2-\sqrt{-2}} = \frac{(4+\sqrt{-2})(2+\sqrt{-2})}{(2-\sqrt{-2})(2+\sqrt{-2})} = \frac{8+6\sqrt{-2}-2}{4-(-2)}.$$

$$= \frac{6+6\sqrt{-2}}{6} = 1 + \sqrt{-2}.$$

Ex. 3. Prove that $\sqrt{4+3\sqrt{-20}} + \sqrt{4-3\sqrt{-20}} = 6$.

Let $\sqrt{4+3\sqrt{-20}} = \sqrt{x} + \sqrt{y}$, $\therefore \sqrt{4-3\sqrt{-20}} = \sqrt{x} - \sqrt{y}$;

and $$\left.\begin{aligned} \sqrt{16+180} = x - y,\ \therefore 14 &= x - y \\ \text{but } 4 &= x + y \end{aligned}\right\}$$

whence $x = 9$, $y = -5$, and $\sqrt{4 \pm 3\sqrt{-20}} = 3 \pm \sqrt{-5}$, from which the required result is evident.

Ex. 4. Find two numbers whose difference is 10, and product $= \frac{1}{3}$ the square of their sum.

Let x = least of the two, and $x + 10$ = the greater; then $x(x+10) = \frac{1}{3}(2x+10)^2$, whence $x = -5 \pm 5\sqrt{-3}$.

Hence we infer that there is no possible solution to the given question, which in fact amounts to asking two numbers x and y, such that

$$xy = \tfrac{1}{3}(x+y)^2, \text{ or } 3xy = x^2 + 2xy + y^2, \text{ or } xy = x^2 + y^2,$$

and this we know from (55) to be impossible.

Whenever, then, in solving a problem, we arrive at imaginary results, we may infer that there is some condition involved in the question, which renders its solution impracticable. If the roots are merely *surd*, and not imaginary, they give, of course, a real solution.

EXAMPLES.

Reduce

531. $2\sqrt{-4}+3\sqrt{-9}+5\sqrt{-16}$; $2\sqrt{-12}+\dfrac{1}{\sqrt{-3}}-3\sqrt{-4\tfrac{11}{16}}$.

532. $(1+\sqrt{-1})^2+(1+\sqrt{-1})^3+(1+\sqrt{-1})^4$;

$$(x+1)\left(x-\frac{1+\sqrt{-3}}{2}\right)\left(x-\frac{1-\sqrt{-3}}{2}\right).$$

533. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$; $\dfrac{3-3\sqrt{-1}}{2+2\sqrt{-1}}$; and

$$\left\{\frac{1+\sqrt{-2}}{1-\sqrt{-2}}+\frac{1-\sqrt{-2}}{1+\sqrt{-2}}\right\}\times\left\{\frac{1+\sqrt{-2}}{1-\sqrt{-2}}-\frac{1-\sqrt{-2}}{1+\sqrt{-2}}\right\}.$$

534. $(x-a\sqrt{-1})(x+a\sqrt{-1})\left\{x+\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}\times$

$$\left\{x+\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}\left\{x-\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}\left\{x-\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}.$$

535. Find the 4th power of $-\sqrt{(-2\sqrt{-3})}$, and the square of $\sqrt{x+\sqrt{-y^2}}+\sqrt{x-\sqrt{-y^2}}$.

536. Find the square roots of $31+12\sqrt{-5}$, and $-24\sqrt{-1}-7$; and the fourth root of -4.

CHAPTER VIII.

THE BINOMIAL THEOREM.

120. The *Binomial Theorem* is a formula by which any binomial may be raised to any given power, without going through the ordinary process of involution. The simplest form in which this formula can be given, and to which all cases may be reduced, is

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.;$$

which, putting C_1, C_2, &c. for the coeff^s of x, x^2, &c. we may write

$$(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \&c.,$$

where C_r = coeff. of $x^r = \dfrac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}$.

N.B. C_r is the coeff. of the $(r+1)^{th}$ term of the expansion.

121. *To prove the Binomial Theorem, when the index is a* POSITIVE INTEGER.

It is found by involution, that

$$(1+x)^2 = 1 + 2x + x^2,$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4, \&c.$$

in which expansions it will be found that the coeff^s are formed according to the above law: we shall now shew that, *supposing* the law to be true for the coeff^s of any one power of $1+x$, it will also be for those of the next higher power; *i.e.* if it be true for $(1+x)^{n-1}$, it will also be for $(1+x)^n$.

Suppose then

$$(1+x)^{n-1} = 1 + B_1x + B_2x^2 + B_3x^3 + \&c. \ldots\ldots (1),$$

where B_1, B_2, &c. are formed like C_1, C_2, &c., by writing $n-1$ in the place of n; so that

$$B_1 = \frac{n-1}{1},\ B_2 = \frac{(n-1)(n-2)}{1\,.\,2},\ B_3 = \frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3},\ \&c.$$

Multiplying both sides of (1) by $1+x$, we have

$$\begin{array}{lllll} (1+x)^n = 1 & + B_1x & + B_2x^2 & + B_3x^3 & + \&c. \\ & + x & + B_1x^2 & + B_2x^3 & + \&c. \\ \hline \multicolumn{5}{l}{= 1 + (1+B_1)x + (B_1+B_2)x^2 + (B_2+B_3)x^3 + \&c.} \end{array}$$

Now $\quad 1 + B_1 = 1 + (n-1) = n = C_1,$

$$B_1 + B_2 = (n-1) + \frac{(n-1)(n-2)}{1\,.\,2} = (n-1)\left(1 + \frac{n-2}{2}\right)$$

$$= (n-1)\frac{n}{2} = \frac{n(n-1)}{1\,.\,2} = C_2,$$

$$B_2 + B_3 = \frac{(n-1)(n-2)}{1\,.\,2}\left(1 + \frac{n-3}{3}\right) = \frac{(n-1)(n-2)}{1\,.\,2}\left(\frac{n}{3}\right)$$

$$= \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} = C_3,\ \&c.$$

and so $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \&c.$

If then this formula be true for any one power of $1+x$, it is true for the next higher power: but it is true for the second, third, fourth, &c. powers; therefore, for the fifth, and therefore, for the sixth, &c., and so is generally true, when the index is a *positive integer.*

122. *There are only* n + 1 *terms in the expansion of* (1 + x)^n, *when the index is a positive integer.*

Since (120 Cor). $C_r = \dfrac{n(n-1)\ldots\ldots(n-r+1)}{1\,.\,2\,\ldots\ldots\,r}$ = coeff. of

$(r+1)^{\text{th}}$ term, if r be such that the last factor of the num$^{\text{r}}$, $n - r + 1 = 0$, then this and all the *following* terms (all of which would involve this factor) will vanish, or the series will have ended with the preceding or r^{th} term. Now if $n - r + 1 = 0$, then $r = n + 1$; and the series will have ended with the $(n + 1)^{\text{th}}$ term.

123. *In the expansion of* $(1 + x)^n$, *the coeff^s of terms, equally distant from the beginning and end, are the* SAME, *when the index is a positive integer.*

The $(r + 1)^{\text{th}}$ term from the end (having r *after* it) will be the $\{(n + 1) - r\}^{\text{th}}$ or $(n - r + 1)^{\text{th}}$ from the beginning: its coeff. will therefore (120) be C_{n-r}, and we find, by writing $n - r$ for r in the expression for C_r, that $C_{n-r} = \dfrac{n(n-1)\ldots\ldots(r+1)}{1\,.\,2\ldots\ldots(n-r)}$.

Now, in the num^r of this fraction, the factors regularly descend, and in the den^r ascend; hence, in the num^r, (supposing C_r nearer the beginning than C_{n-r}, and, therefore, $n - r + 1 > r + 1$) the factor $n - r + 1$ will come in between n and $r + 1$, and next *after* it will come $n - r$; and in the den^r the factor $r + 1$ will come in between 1 and $n - r$, and next *before* it will come r; so that we may write

$$C_{n-r} = \frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots(r+1)}{1\,.\,2\ldots\ldots r(r+1)\ldots\ldots(n-r)},$$

or striking out the common factors in the num^r and den^r,

$$= \frac{n(n-1)\ldots\ldots(n-r+1)}{1\,.\,2\ldots\ldots r} = C_r;$$

i. e. coeff. of $(r + 1)^{\text{th}}$ term from end = coeff. of $(r + 1)^{\text{th}}$ term from beginning.

124. The number of terms, $n + 1$, being *odd* or *even* as n is *even* or *odd*, it follows that, if n be even, there will be *one* middle term, but if odd, *two* equal middle terms (123), on each side of which the *same* coeff^s will occur

in order. When, therefore, in expanding a binomial, we have passed the middle term or terms, we shall find all the coeff^s repeating themselves; and, instead of calculating those of the remaining terms, we may write down, in inverted order, the coeff^s already found, as in the following examples.

Ex. 1. $(1+x)^4 = 1 + \frac{4}{1}x + \frac{4.3}{1.2}x^2 + \&c.$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4.$$

Ex. 2. $(1+x)^7 = 1 + \frac{7}{1}x + \frac{7.6}{1.2}x^2 + \frac{7.6.5}{1.2.3}x^3 + \&c.$

$$= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$

Ex. 3. $(1+2x)^5 = 1 + \frac{5}{1}(2x) + \frac{5.4}{1.2}(2x)^2 + \&c.$

$$= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + 5(16x^4) + 32x^5$$
$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

EXAMPLES.

537. $(1+x)^3$.
538. $(1+x)^6$.
539. $(1+x)^8$.
540. $(1+x)^{12}$.
541. $(1+2x)^4$.
542. $(1+4x)^5$.
543. $(1+3x)^6$.
544. $(1+2x)^8$.
545. $\left(1+\frac{x}{2}\right)^5$.
546. $\left(1+\frac{x}{4}\right)^4$.

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125. *To prove the Binomial Theorem, when the index is* FRACTIONAL *or* NEGATIVE.

Let the series $1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \&c.$, *whatever* be the value of m, be denoted by the symbol $f(m)$. Now, when m is a + integer, we know that this series represents the expansion of $(1+x)^m$, that is, $(1+x)^m = f(m)$, when m is a + integer. We shall now shew that this is the case for *all* values of m.

Since
$$f(m) = 1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \&c.$$

$$\therefore\ f(n) = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.$$

and
$$\begin{aligned} f(m)\times f(n) = 1 &+ \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \&c. \\ &+ \frac{n}{1}x + \quad mn \quad x^2 + \&c. \\ &\qquad\quad + \frac{n(n-1)}{1.2}x^2 + \&c. \\ = 1 &+ ax + \quad bx^2 \quad + \&c. \end{aligned}$$

where we use a, b, &c. to denote the coeff^s found by addition, of x, x^2, &c., so that

$$a = m + n, \quad b = \frac{m(m-1)}{1.2} + mn + \frac{n(n-1)}{1.2}, \quad \&c.$$

Now, b, c, &c. might be reduced to much simpler forms than these, but the process would be tedious: we may find them, however, immediately, by the following consideration. Since the above multiplication does not at all depend upon the actual values of m and n, we should still have, by the addition, the same *values* as above for a, b, &c., whether m and n stand for + or −, integral or fractional, quantities, though these may admit, as has been said, of being expressed in other and simpler *forms*. If then we can ascertain by any means what these forms are when m and n are *any* + *integers*, they will be the same when m and n have *any* values.

Now, when m and n are + integers, we know that

$$f(m)=(1+x)^m, \quad f(n)=(1+x)^n,$$

$$\text{and } \therefore f(m)\times f(n)=(1+x)^m\times(1+x)^n=(1+x)^{m+n};$$

but since $m+n$ is a + integer, we have also

$$(1+x)^{m+n}=1+\frac{m+n}{1}x+\frac{(m+n)(m+n-1)}{1.2}x^2+\&c.$$

$$\text{that is, } =f(m+n);$$

and, therefore, *whatever* be the values of m and n, we have

$$f(m)\times f(n)=f(m+n),$$

$$\text{or } \quad =1+\frac{m+n}{1}x+\frac{(m+n)(m+n-1)}{1.2}x^2+\&c.$$

where the *values* of a, b, &c. will be found the same as before, though their *forms* (except that of a) are more simple. Hence also

$$f(m)\times f(n)\times f(p)=f(m+n)\times f(p)=f(m+n+p);$$

and so for any number of such factors.

Now, first, let there be n factors, each $=f\left(\frac{m}{n}\right)$, where m and n are + integers;

$$\text{then } f\left(\frac{m}{n}\right)\times f\left(\frac{m}{n}\right)\times f\left(\frac{m}{n}\right)\times \&c. \ n \text{ factors}$$

$$=f\left(\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\&c. \ n \text{ terms}\right),$$

$$\text{or } \left\{f\left(\frac{m}{n}\right)\right\}^n=f\left(\frac{mn}{n}\right)=f(m)=(1+x)^m,$$

since m is a + integer;

$\therefore$ taking the n^{th} root on both sides, $(1+x)^{\frac{m}{n}}=f\left(\frac{m}{n}\right)$.

Hence $(1+x)^m=f(m)$, when the index is *positive*, whether *integral* or *fractional*.

Again, let $n = -m$, where m is *positive*, but may be integral or fractional; then

$$f(m) \times f(-m) = f(m - m) = f(0) = 1,$$

(since the series becomes $= 1$, if we put 0 for m in it);

$$\therefore f(-m) = \frac{1}{f(m)} = \frac{1}{(1+x)^m}, \text{ since } m \text{ is } \textit{positive},$$

$$= (1+x)^{-m}, \text{ by Theory of Indices.}$$

Hence $(1+x)^{-m} = f(-m)$,

when the index is *negative*, whether *integral* or *fractional.*

Whatever, then, be the value of the index, we have, as stated in (120),

$$(1+x)^n = f(n) = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.$$

126. We have seen (122), that when the index is a + integer, this series will stop after $n+1$ terms; when fractional or negative, it will never terminate, but consist of an infinite number of terms, since we cannot then have $n - r + 1 = 0$, for any value of r.

Ex. 1.

$$(1+x)^{-2} = 1 + \frac{-2}{1}x + \frac{-2(-2-1)}{1.2}x^2 + \frac{-2(-2-1)(-2-2)}{1.2.3}x^3 + \&c.$$

$$= 1 - \frac{2}{1}x + \frac{2.3}{1.2}x^2 - \frac{2.3.4}{1.2.3}x^3 + \&c. = 1 - 2x + 3x^2 - 4x^3 + \&c.$$

In this example there is some trouble in simplifying coeff^s, and getting rid of superfluous signs; to save this in practice, it will be useful to remember the result of the following general example.

Ex. 2.

$$(1+x)^{-n} = 1 + \frac{-n}{1}x + \frac{-n(-n-1)}{1.2}x^2 + \frac{-n(-n-1)(-n-2)}{1.2.3}x^3 + \&c.$$

$$= 1 - \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + \&c. (1).$$

Ex. 3. $(1-x)^{-3} = 1 + \frac{3}{1}x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \&c.$

$$= 1 + 3x + 6x^2 + 10x^3 + \&c.$$

Ex. 4. $(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x^3 + \&c.$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{1.2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1.2.3}x^3 + \&c.$$

$$= 1 + \frac{1}{2}x - \frac{1}{1.2.2^2}x^2 + \frac{3}{1.2.3.2^3}x^3 - \&c.$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \&c.$$

Here also notice the following two general results.

Ex. 5.

$$(1+x)^{\frac{p}{q}} = 1 + \frac{\frac{p}{q}}{1}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{1.2.3}x^3 + \&c.$$

$$= 1 + \frac{p}{q}x + \frac{\frac{p}{q}.\frac{p-q}{q}}{1.2}x^2 + \frac{\frac{p}{q}.\frac{p-q}{q}.\frac{p-2q}{q}}{1.2.3}x^3 + \&c.$$

$$= 1 + \frac{p}{q}x + \frac{p(p-q)}{1.2.q^2}x^2 + \frac{p(p-q)(p-2q)}{1.2.3.q^3}x^3 + \&c. \ldots (2)$$

So also

$$(1+x)^{-\frac{p}{q}} = 1 - \frac{p}{q}x + \frac{p(p+q)}{1.2.q^2}x^2 - \frac{p(p+q)(p+2q)}{1.2.3.q^3}x^3 + \&c.. \ (3)$$

Ex. 6. $(1+x)^{-\frac{2}{3}} = 1 - \frac{2}{3}x + \frac{2.5}{1.2.3^2}x^2 - \frac{2.5.8}{1.2.3.3^3}x^3 + \&c.$

$$= 1 - \tfrac{2}{3}x + \tfrac{5}{9}x^2 - \tfrac{40}{81}x^3 + \&c.$$

127. Since $(1-x)^n = \{1 + (-x)\}^n$

$$= 1 + C_1(-x) + C_2(-x)^2 + C_3(-x)^3 + \&c.$$

$$= 1 - C_1 x + C_2 x^2 - C_3 x^3 + \&c.$$

the expansion for $(1-x)^n$ differs from that for $(1+x)^n$, only in having the terms alternately + and −. Hence

$$(1-x)^{-n} = 1 + \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \&c. \ldots (4),$$

$$(1-x)^{\frac{p}{q}} = 1 - \frac{p}{q}x + \frac{p(p-q)}{1.2.q^2}x^2 - \frac{p(p-q)(p-2q)}{1.2.3.q^3}x^3 + \&c. \ldots (5),$$

$$(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{q}x + \frac{p(p+q)}{1.2.q^2}x^2 + \frac{p(p+q)(p+2q)}{1.2.3.q^3}x^3 + \&c. \ldots (6).$$

EXAMPLES.

547. $(1+x)^{-1}$.

548. $(1-3x)^{-1}$.

549. $(1+3x)^{-2}$.

550. $(1-2x)^{-3}$.

551. $\left(1-\frac{x}{2}\right)^{-2}$.

552. $\left(1+\frac{x}{3}\right)^{-3}$.

553. $(1+2x)^{\frac{1}{2}}$.

554. $(1-3x)^{\frac{2}{3}}$.

555. $(1-x)^{-\frac{3}{2}}$.

556. $(1-x^2)^{\frac{7}{3}}$.

557. $\frac{1}{\sqrt[3]{1-x}}$.

558. $\frac{1}{\sqrt{1-x^2}}$.

128. Since $a \pm x = a\left(1 \pm \frac{x}{a}\right)$,

$$\therefore (a \pm x)^n = a^n\left(1 \pm \frac{x}{a}\right)^n = a^n\left\{1 \pm C_1\left(\frac{x}{a}\right) + C_2\left(\frac{x}{a}\right)^2 \pm C_3\left(\frac{x}{a}\right)^3 + \&c.\right\}$$

$$= a^n \pm C_1 a^{n-1}x + C_2 a^{n-2}x^2 \pm C_3 a^{n-3}x^3 + \&c.$$

where we observe that the index of a, beginning from n, is diminished by unity, and the index of x, beginning from 0, is increased by unity in each succeeding term.

Ex. 1. $(2a+x)^5 = (2a)^5 + \frac{5}{1}(2a)^4x + \frac{5.4}{1.2}(2a)^3x^2 + \&c.$

$$= 32a^5 + 5(16a^4)x + 10(8a^3)x^2 + 10(4a^2)x^3 + 5(2a)x^4 + x^5$$

$$= 32a^5 + 80a^4x + 80a^3x^2 + 40a^2x^3 + 10ax^4 + x^5.$$

Ex. 2. $(a + 2x)^{-4} = a^{-4}\left(1 + \frac{2x}{a}\right)^{-4}$

$$= a^{-4}\left\{1 - \frac{4}{1}\left(\frac{2x}{a}\right) + \frac{4.5}{1.2}\left(\frac{2x}{a}\right)^2 - \frac{4.5.6}{1.2.3}\left(\frac{2x}{a}\right)^3 + \&c.\right\}$$

$$= a^{-4} - 8a^{-5}x + 40a^{-6}x^2 - 160a^{-7}x^3 + \&c.$$

Ex. 3. $(a - 2x)^{-\frac{3}{2}} = a^{-\frac{3}{2}}\left(1 - \frac{2x}{a}\right)^{-\frac{3}{2}}$

$$= a^{-\frac{3}{2}}\left\{1 + \frac{3}{2}\left(\frac{2x}{a}\right) + \frac{3.5}{1.2.2^2}\left(\frac{2x}{a}\right)^2 + \frac{3.5.7}{1.2.3.2^3}\left(\frac{2x}{a}\right)^3 + \&c.\right\}$$

$$= a^{-\frac{3}{2}} + 3a^{-\frac{5}{2}}x + \tfrac{15}{2}a^{-\frac{7}{2}}x^2 + \tfrac{35}{2}\,a^{-\frac{9}{2}}x^3 + \&c.$$

129. A *trinomial* or *multinomial* may be expanded by the binomial theorem, as in the following examples.

Ex. 1. $(a + b + c)^3 = \{a + (b + c)\}^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2$ $+ (b + c)^3 = a^3 + 3a^2(b + c) + 3a(b^2 + 2bc + c^2) + b^3 + 3b^2c + 3bc^2 + c^3$ $= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.$

Ex. 2. $(1 - x + x^2)^4 = \{(1 - x) + x^2\}^4$

$$= (1 - x)^4 + 4(1 - x)^3x^2 + 6(1 - x)^2x^4 + 4(1 - x)x^6 + x^8$$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2(1 - 3x + 3x^2 - x^3)$$

$$+ 6x^4(1 - 2x + x^2) + 4x^6(1 - x) + x^8$$

$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8.$$

EXAMPLES.

559. $(a + x)^6$.
560. $(a - x)^9$.
561. $(2a - 3x)^5$.
562. $(3x + y)^7$.
563. $(2 - x)^{-2}$.
564. $(3 - 2x)^{-3}$.
565. $(a + bx)^{-1}$
566. $(a - b^2x)^{-2}$.
567. $(a^{-\frac{1}{3}} - b^{\frac{1}{3}})^{-6}$
568. $(a^{-\frac{1}{3}} + b^{-\frac{1}{3}})^{-3}$.

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569. $(a - x)^{\frac{1}{3}}$.   570. $(a^2 - x^2)^{\frac{1}{5}}$.

571. $(a^5 - x^5)^{-\frac{1}{5}}$.   572. $(a^2 - x^2)^{\frac{2}{3}}$.

573. $(a^2 - x^2)^{-\frac{3}{4}}$.   574. $(ax - x^2)^{-\frac{1}{3}}$.

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575. $(1 - x + x^2)^3$.   576. $(1 + x + x^2)^4$.
577. $(1 + x - x^2)^5$.   578. $(1 - 2x + x^2)^3$.
579. $(1 + 2x + 3x^2)^4$.   580. $(a - b - c)^3$.
581. $(a - 2b + c)^4$.   582. $(a + 2b - c)^4$.

130. The sum of all the coeff^{s} of $(1 + x)^n = 2^n$; and sum of even coeff^{s} = sum of odd coeff^{s} = 2^{n-1}.

For since $(1 \pm x)^n = 1 \pm C_1 x + C_2 x^2 \pm C_3 x^3 + \text{\&c.}$,

$$\therefore\ (1 + 1)^n = 2^n = 1 + C_1 + C_2 + C_3 + \text{\&c.} \ldots (1),$$

$$\text{and } (1 - 1)^n = 0 = 1 - C_1 + C_2 - C_3 + \text{\&c.} \ldots (2);$$

hence (1) sum of *all* coeff^{s} of $(1 + x)^n = 2^n$,
and (2) $1 + C_2 + \text{\&c.} = C_1 + C_3 + \text{\&c.}$,
or sum of even coeff^{s} = sum of odd,

$$\text{and } \therefore \text{ each} = \tfrac{1}{2}(2^n) = 2^{n-1}.$$

CHAPTER IX.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

131. Quantities are said to be in *Arithmetical Progression,* when they increase or decrease by a common *difference:* thus

$1, 3, 5, 7$, &c., $8, 4, 0, -4$, &c., $a, a+d, a+2d, a+3d$, &c.,

are in AR. PROG., the common differences being $2, -4, d$, respectively.

The first is therefore an instance of quantities *increasing,* the second of quantities *decreasing* in AR. PROG., the common difference being *positive* in the former, and *negative* in the latter; in the third, the series will be increasing or decreasing, as d represents a + or − quantity.

132. *If* a *be the first term, and* l *the* n^th^ *term of an* AR. *series, whose common difference is* d, *then*

$$l = a + (n - 1)\, d.$$

Here the series will be $a, a + d, a + 2d, a + 3d$, &c., where the coeff. of d in any term is just *less by one* than the No. of the term; thus in the 2nd term we have d, *i.e.* $1d$, in the 3rd, $2d$, in the 4th, $3d$, &c., and so in the n^{th} term we shall have $(n - 1)\, d$; hence $l = a + (n - 1)\, d$.

N.B. In any AR. series, the common difference may be found by *subtracting any term from the term following.*

Ex. 1. Find the 10th term of the series 1, 5, 9, &c.

Here $a = 1, d = 4, n = 10$;

$$\therefore l = 1 + (10 - 1)\, 4 = 1 + 9 \times 4 = 37.$$

Ex. 2. Find the 9th term of the series $7, 5\frac{1}{2}, 4$, &c.

Here $a = 7, d = -\frac{3}{2}, n = 9$;

$$\therefore\ l = 7 + (9 - 1) \times -\tfrac{3}{2} = 7 - 8 \times \tfrac{3}{2} = -5.$$

Ex. 3. Find the 13th term of the series $-48, -44, -40$, &c.

Here $a = -48, d = 4, n = 13$;

$$\therefore\ l = -48 + (13 - 1)\,4 = -48 + 12 \times 4 = 0.$$

EXAMPLES.

583. Find the 9th term of the series 1, 3, 5, &c.
584. Find the 13th term of the series 3, 9, 15, &c.
585. Find the 100th term of the series 1, 9, 17, &c.
586. Find the 8th term of the series $-5, -3, -1$, &c.
587. Find the 10th term of the series $4, -3, -10$, &c.
588. Find the 20th term of the series 7, 5, 3, &c.
589. Find the 25th term of the series $8, 8\frac{1}{4}, 8\frac{1}{2}$, &c.
590. Find the 7th term of the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, &c.
591. Find the 6th term of the series $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}$, &c.
592. Find the 20th term of the series $\frac{1}{2}, \frac{1}{6}, -\frac{1}{6}$, &c.
593. Find the 13th term of the series $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}$, &c.
594. Find the 29th term of the series $\frac{1}{2}, -1, -\frac{5}{2}$, &c.

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133. *If* S *be the sum of* n *terms of an* AR. *series, whose first term is* a, nth *term* l, *and common difference* d, *then* $S = \{2a + (n - 1)\,d\}\,\frac{n}{2}$, *or* $S = (a + l)\,\frac{n}{2}$.

For $S = a + (a + d) + \&c. + \{a + (n - 2)\,d\} + \{a + (n - 1)d\}$; and, again, reversing the order of the terms, we have

$$S = \{a + (n - 1)\,d\} + \{a + (n - 2)\,d\} + \&c. + (a + d) + a;$$

$\therefore\ 2S = \{2a + (n - 1)d\} + \{2a + (n - 1)\,d\} + \&c.$ by addition, the same quantity $\{2a + (n - 1)\,d\}$ being the sum of each pair of terms; now there are n such pairs, and therefore, on the whole, we have

$$2S = \{2a + (n-1)d\}\, n, \text{ and } S = \{2a + (n-1)d\}\, \frac{n}{2};$$

but $a + l = a + \{a + (n-1)d\} = \{2a + (n-1)d\}$; $\therefore$ also $S = (a+l)\frac{n}{2}$.

Ex. 1. Find the sum of 10 terms of $2 + 5 + 8 + \&c.$

Here $a = 2$, $d = 3$, $n = 10$;

$$\therefore S = \{2a + (n-1)d\}\frac{n}{2} = \{4 + 9 \times 3\}\frac{10}{2} = (4 + 27)\, 5 = 155.$$

Ex. 2. Find the sum of 7 terms of $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \&c.$

Here $a = \frac{1}{2}$, $d = -\frac{1}{6}$, $n = 7$;

$$\therefore S = (1 + 6 \times -\tfrac{1}{6})\,\tfrac{7}{2} = (1 - 1)\,\tfrac{7}{2} = 0.$$

In this case the series, continued, is

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, 0, -\tfrac{1}{6}, -\tfrac{1}{3}, -\tfrac{1}{2}, \&c.$$

where the first 7 terms together amount to zero.

EXAMPLES.

Find the sum

595. Of $2 + 4 + 6 + \&c.$ to 16 terms.

596. Of $1 + 3 + 5 + \&c.$ to 20 terms.

597. Of $3 + 9 + 15 + \&c.$ to 11 terms.

598. Of $1 + 8 + 15 + \&c.$ to 100 terms.

599. Of $-5 - 3 - 1 - \&c.$ to 8 terms.

600. Of $1 + \frac{6}{7} + \frac{5}{7} + \&c.$ to 15 terms.

601. Of $\frac{2}{3} + \frac{7}{15} + \frac{4}{15} + \&c.$ to 21 terms.

602. Of $4 - 3 - 10 - \&c.$ to 10 terms.

603. Of $\frac{1}{2} + \frac{3}{4} + 1 + \&c.$ to 10 terms.

604. Of $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} - \&c.$ to 13 terms.

605. Of $1 + 2\frac{2}{3} + 4\frac{1}{3} + \&c.$ to 20 terms.

606. Of $\frac{3}{5} - \frac{14}{10} - \frac{51}{15} - \&c.$ to 10 terms.

134. From the equation $l = a + (n - 1)d$, we can not only find l, when a, d, n, are given, but also a, when l, n, d, are given, $\{\text{since } a = l - (n - 1)d\}$, and, in fact, *any one* of the quantities a, d, l, n, when the other *three* are given.

The same remark applies to the equations

$$S = \{2a + (n - 1)d\}\frac{n}{2}, \text{ and } S = (a + l)\frac{n}{2},$$

when any 3, of the 4 quantities they contain, are given.

These equations may also be used to solve many problems in AR. PROG. as in the following Examples.

Ex. 1. The first term of an AR. series is 3, the 13th term, 55; find the common difference.

Since $l = a + (n - 1)d$, and here $l = 55$, $a = 3$, $n = 13$, we have $55 = 3 + 12d$, and $\therefore d = 4\frac{1}{3}$.

Ex. 2. What No. of terms of the series 10, 8, 6, &c. must be taken to make 30? and what No. to make 28?

$$S = \{2a + (n - 1)d\}\frac{n}{2}.$$

In the first case,

$$S = 30,\ a = 10,\ d = -2,\ \ \therefore 30 = \{20 - 2(n - 1)\}\frac{n}{2},$$

and the roots of this quadratic are 5 and 6, either of which satisfies the question, since the *sixth* term of the series is zero.

In the second case, $S = 28$, $a = 10$, $d = -2$; and the roots of the resulting quadratic are 4 and 7, either of which also satisfies the question, since the 5th, 6th, and 7th terms of the series, 2, 0, -2, are together = zero.

Ex. 3. Insert 3 AR. *means* (or intermediate terms) between 6 and 26.

Here we have to find three numbers between 6 and 26, so that the five may be in AR. PROG. This case then reduces itself to finding d, when $a = 6$, $l = 26$, and $n = 5$; we have, therefore, by the formula $l = a + (n - 1)d$, $26 = 6 + 4d$, whence $d = 5$, and the means required are 11, 16, 21.

Ex. 4. The sum of three numbers in AR. PROG. is 21, and the sum of their squares, 179; find them.

Let $a - d$, a, $a + d$, represent the three numbers, (which is often a convenient assumption in problems of this kind);
then $(a-d)+a+(a+d)=21$, and $(a-d)^2+a^2+(a+d)^2=179$, from which equations $a = 7$, $d = \pm 4$, and $\therefore$ the Nos. required are 3, 7, 11.

EXAMPLES.

607. The first term of an AR. series is 2, the common difference 7, and the last term 79; find the number of terms.

608. The sum of 15 terms of an arithmetic series is 600, and the common difference is 5; find the first term.

609. The first term is $13\frac{1}{15}$, the common difference $-\frac{2}{3}$, and the last term $\frac{2}{5}$; find the number of terms.

610. The sum of 11 terms is $14\frac{4}{7}$, and the common difference is $\frac{2}{7}$; find the first term.

611. Insert 4 AR. means between 2 and 17.

612. Insert 4 AR. means between 2 and -18.

613. Insert 9 AR. means between 3 and 9.

614. Insert 7 AR. means between -13 and 3.

615. Insert 10 AR. means between -7 and 114.

616. Insert 8 AR. means between -3 and $-\frac{3}{4}$.

617. Insert 9 AR. means between $-2\frac{3}{4}$ and $4\frac{3}{4}$.

618. Insert 9 AR. means between $-3\frac{2}{5}$ and $2\frac{3}{5}$.

619. There are 3 Nos. in AR. PROG., whose sum is 21, and the sum of the first and second = $\frac{3}{4}$ that of the second and third; find them.

620. There are 3 Nos. in AR. PROG., whose sum is 10, and the product of the second and third $33\frac{1}{3}$; find them.

621. Find 3 Nos. in AR. PROG., such that, the common difference being 1, the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

622. The first term is $n^2 - n + 1$, the common difference 2; find the sum of n terms.

623. A debt can be discharged in a year by paying one shilling the first week, three the second, five the third, &c.; required the last payment and the amount of the debt.

624. One hundred stones being placed on the ground at the distance of a yard from one another, how far will a person travel, who shall bring them, one by one, to a basket, placed at the distance of a yard from the first stone?

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135. Quantities are said to be in *Geometrical Progression*, when they increase or decrease by a common *factor:* thus,

$$1, 3, 9, 27, \&c. \quad 16, 4, 1, \tfrac{1}{4}, \&c. \quad \tfrac{1}{3}, -\tfrac{4}{15}, \tfrac{16}{75}, -\tfrac{64}{375}, \&c.$$
$$a, ar, ar^2, ar^3, \&c.$$

are in GEOM. PROG., the common factors or *ratios* (as they are called) being $3, \frac{1}{4}, -\frac{4}{5}, r$, respectively.

The first is an instance of an *increasing* series, the second and third of *decreasing* series, the common ratio being an *integer* in the former, and a *proper fraction* in the latter two: in the fourth, the series will be increasing or decreasing, as r is an integer or a proper fraction.

136. *If* a *be the first term and* l *the* n^th^ *term of a geom. series, whose common ratio is* r, *then* $l = ar^{n-1}$.

Here the series will be a, ar, ar^2, ar^3, &c., where the index of r in any term is just *less by one* than the number of the term: thus, in the 2nd term we have r, i.e. r^1, in the 3rd, r^2, in the 4th, r^3, &c., and so in the nth term we shall have r^{n-1}; hence $l = ar^{n-1}$.

N.B. In any geom. series, the common ratio may be found by *dividing any term by the term preceding.*

Ex. 1. Find the 6th term of the series 1, 2, 4, &c.

Here $a = 1, \quad r = 2, \quad n = 6;$

$$\therefore l = 1 \times 2^{6-1} = 1 \times 2^5 = 1 \times 32 = 32.$$

Ex. 2. Find the 8th term of the series 81, −27, 9, &c.

Here $a = 81, \quad r = -\frac{1}{3}, \quad n = 8;$

$$\therefore l = 81 \times \left(-\frac{1}{3}\right)^7 = 3^4 \times -\frac{1}{3^7} = -\frac{1}{3^3} = -\frac{1}{27}.$$

EXAMPLES.

625. Find the 4th term of the series 1, 3, 9, &c.
626. Find the 7th term of the series 1, 4, 16, &c.
627. Find the 10th term of the series $1, \frac{3}{2}, \frac{9}{4}$, &c.
628. Find the 8th term of the series $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}$, &c.
629. Find the 5th term of the series 4, −12, 36, &c.
630. Find the 8th term of the series 9, −6, 4, &c.
631. Find the 5th term of the series $3\frac{3}{8}, 2\frac{1}{4}, 1\frac{1}{2}$, &c.
632. Find the 6th term of the series $3, \frac{1}{2}, \frac{1}{12}$, &c.
633. Find the 4th term of the series $1\frac{2}{5}, 3\frac{1}{2}, 8\frac{3}{4}$, &c.
634. Find the 5th term of the series $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{9}$, &c.
635. Find the 6th term of the series $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}$, &c.
636. Find the 5th term of the series $3, -4\frac{1}{2}, 6\frac{3}{4}$, &c.

137. *If* S *be the sum of* n *terms of a geom. series, whose first term is* a, $\mathrm{n^{th}}$ *term* l, *and common ratio* r, *then*

$$\mathrm{S} = \frac{\mathrm{ar^n - a}}{\mathrm{r - 1}} = \mathrm{a} \,.\, \frac{\mathrm{r^n - 1}}{\mathrm{r - 1}}, \quad \textit{or} \quad \mathrm{S} = \frac{\mathrm{rl - a}}{\mathrm{r - 1}}.$$

For $$S = a + ar + ar^2 + \&\text{c.} + ar^{n-1},$$

$$\therefore\ rS = ar + ar^2 + ar^3 + \&\text{c.} + ar^n;$$

subtracting, $rS - S = ar^n - a$, all the other terms destroying one another;

hence $(r-1)S = ar^n - a = a\,(r^n - 1)$, and $S = \dfrac{ar^n - a}{r - 1} = a \,.\, \dfrac{r^n - 1}{r - 1}$;

but $rl - a = r \times ar^{n-1} - a = ar^n - a$; $\therefore$ also $S = \dfrac{rl - a}{r - 1}$.

Ex. 1. Find the sum of $1 + 3 + 9 + \&\text{c.}$ to 5 terms.

Here $$a = 1, \quad r = 3, \quad n = 5;$$

$$\therefore\ S = a \,.\, \frac{r^n - 1}{r - 1} = 1 \times \frac{3^5 - 1}{3 - 1} = \frac{243 - 1}{2} = \frac{242}{2} = 121.$$

Ex. 2. Find the sum of $3 - 6 + 12 - \&\text{c.}$ to 6 terms.

Here $$a = 3, \quad r = -2, \quad n = 6;$$

$$\therefore\ S = 3 \,.\, \frac{(-2)^6 - 1}{-2 - 1} = 3 \,.\, \frac{64 - 1}{-3} = -63.$$

Ex. 3. Find the sum of $1 - \frac{4}{3} + \frac{16}{9} - \&\text{c.}$ to 4 terms.

Here $$a = 1, \quad r = -\tfrac{4}{3}, \quad n = 4;$$

$$\therefore\ S = 1 \times \frac{\left(-\dfrac{4}{3}\right)^4 - 1}{-\dfrac{4}{3} - 1} = \frac{\dfrac{4^4}{3^4} - 1}{-\dfrac{4}{3} - 1} = \frac{\dfrac{4^4 - 3^4}{3^4}}{-\dfrac{7}{3}} = -\frac{3}{7} \times \frac{256 - 81}{3^4} = -\frac{175}{7.3^3}$$

$$= -\frac{25}{27}.$$

Ex. 4. Find the sum of $2\frac{1}{2} - 1 + \frac{2}{5} - \&c.$ to 5 terms.

Here $a = \frac{5}{2}, \quad r = -\frac{2}{5}, \quad n = 5;$

$$\therefore S = \frac{5}{2} \cdot \frac{\left(-\frac{2}{5}\right)^5 - 1}{-\frac{2}{5} - 1} = \frac{5}{2} \cdot \frac{-\frac{2^5}{5^5} - 1}{-\frac{2}{5} - 1} = \frac{5}{2} \cdot \frac{\frac{2^5 + 5^5}{5^5}}{\frac{7}{5}}$$

$$= \frac{5}{2} \cdot \frac{5}{7} \cdot \frac{32 + 3125}{5^5} = \frac{3157}{14.5^3} = 1\tfrac{201}{250}.$$

EXAMPLES.

637. Sum $1 + 4 + 16 + \&c.$ to 4 terms.
638. Sum $5 + 20 + 80 + \&c.$ to 5 terms.
639. Sum $3 + 6 + 12 + \&c.$ to 6 terms.
640. Sum $2 - 4 + 8 - \&c.$ to 8 terms.
641. Sum $1 - 4 + 16 - \&c.$ to 7 terms.
642. Sum $1 - 2 + 2^2 - \&c.$ to 10 terms.
643. Sum $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \&c.$ to 8 terms.
644. Sum $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \&c.$ to 6 terms.
645. Sum $\frac{3}{2} + 1 + \frac{2}{3} + \&c.$ to 6 terms.
646. Sum $3 - \frac{1}{2} + \frac{1}{12} - \&c.$ to 5 terms.
647. Sum $9 - 6 + 4 - \&c.$ to 9 terms.
648. Sum $100 - 40 + 16 - \&c.$ to 5 terms.

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138. If r be a *proper* fraction, its powers r, r^2, r^3, &c., and therefore the terms ar, ar^2, ar^3, &c. become smaller and smaller, so that if n be *very* great, ar^n will be *very* small; and if n be *infinitely* great, *i.e.* if the series be supposed continued *beyond any conceivable* No. of terms (which is called being continued *ad infinitum*), ar^n will be so small as to be altogether insensible, and may therefore be neglected. Hence, using Σ to denote the sum of such a series, in which $ar^n = 0$, we have $\Sigma = \frac{ar^n - a}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}$.

Ex. 1. Sum $1 + \frac{1}{2} + \frac{1}{4} +$ &c. *ad infinitum.*

Here $a = 1$, $r = \frac{1}{2}$; $\therefore\ \Sigma = \dfrac{1}{1 - \frac{1}{2}} = \dfrac{1}{\frac{1}{2}} = 2$; *i. e.* the more terms we take of this series, the more nearly will their sum = 2, but will never actually = 2, unless we could imagine the series continued without end.

Ex. 2. Sum $2\frac{1}{2} - \frac{1}{2} + \frac{1}{10} -$ &c. *ad infinitum.*

Here $a = 2\frac{1}{2}$, $r = -\frac{1}{5}$; $\therefore\ \Sigma = \dfrac{\frac{5}{2}}{1 - (-\frac{1}{5})} = \dfrac{\frac{5}{2}}{1 + \frac{1}{5}} = \dfrac{\frac{5}{2}}{\frac{6}{5}} = 2\frac{1}{12}$.

EXAMPLES.

Sum *ad infinitum* the following series:

649. $4 + 2 + 1 +$ &c.
650. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$ &c.
651. $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} -$ &c.
652. $\frac{3}{2} - 1 + \frac{2}{3} -$ &c.
653. $1 - \frac{1}{2} + \frac{1}{4} -$ &c.
654. $1 - \frac{2}{5} + \frac{4}{25} -$ &c.
655. $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} +$ &c.
656. $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} +$ &c.
657. $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} +$ &c.
658. $-3\frac{1}{5} + 1\frac{3}{5} - \frac{4}{5} +$ &c.

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139. By means of the equations of GEOM. PROG., we may solve many problems respecting series of this kind. It is not, however, generally easy to find n, when the other quantities are given, because this quantity occurs in the form of an index. The Student may be able to guess at its value in the simple instances we shall here give; but, in other cases, it could only be found by the aid of logarithms.

Ex. 1. The first term of a GEOM. series is 2, the 7th term is $\frac{1}{32}$; find the series.

$l = ar^{n-1}$; and here $a = 2$, $l = \frac{1}{32}$, $n = 7$;

$\therefore\ \frac{1}{32} = 2r^6$, and $r^6 = \frac{1}{64}$, whence $r = \pm\frac{1}{2}$,

and the series is $2, \pm 1, \frac{1}{2}, \pm\frac{1}{4}$, &c.

Ex. 2. Given 6 the second term of a GEOM. series and 54 the fourth, find the first term.

$$\text{Here } 6 = ar,\ 54 = ar^3,\ \therefore \frac{54}{6} = \frac{ar^3}{ar},\ \text{or } 9 = r^2;$$

$$\text{hence } r = \pm 3,\ \text{and } a = \frac{6}{r} = \pm 2.$$

Ex. 3. Insert 3 GEOM. means between 2 and $10\frac{1}{8}$. Here $\frac{81}{8}$ is the 5th term of a series, whose first term is 2;

$\therefore \frac{81}{8} = 2r^4$, and $r^4 = \frac{81}{16}$; whence $r = \pm \frac{3}{2}$,

and the means are ± 3, $4\frac{1}{2}$, $\pm 6\frac{3}{4}$.

EXAMPLES.

659. How many terms of the series 2, -6, 18, &c. must be taken to make -40?

660. The fifth term of a GEOM. series is 8 times the second, and the third term is 12; find the series.

661. The fifth term of a GEOM. series is 4 times the third, and the sum of the first two is -4; find the series.

662. The population of a country increases annually in GEOM. PROG. and in 4 years was raised from 10000 to 14641 souls; by what part of itself was it annually increased?

663. The difference between the first and second of 4 numbers in GEOM. PROG. is 12, and the difference between the third and fourth is 300; find them.

664. Insert 3 GEOM. means between 2 and 32.

665. Insert 3 GEOM. means between $\frac{1}{2}$ and 128.

666. Insert 4 GEOM. means between $-\frac{1}{10}$ and $3\frac{1}{5}$.

667. Insert 4 GEOM. means between $\frac{2}{3}$ and $-5\frac{1}{16}$.

668. The sum of an infinite GEOM. series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

669. The sum of an infinite GEOM. series is 2, and the second term is $-\frac{3}{2}$; find the series.

670. If a, b, be the first two terms of a GEOM. PROG., find the sum of the series *ad infinitum*.

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140. Quantities are said to be in *Harmonical Progression,* when their reciprocals are in AR. PROG; thus since 1, 3, 5, 7, &c., $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{5}{4}$, &c. are in AR. PROG., their reciprocals

1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c., 4, $-$ 4, $-\frac{4}{3}$, $-\frac{4}{5}$, &c. are in HARM. PROG.

We cannot find the sum of *any* No. of terms of an HARM., as we have of an AR. and GEOM. series; but many problems with respect to such series may be solved by *inverting* the terms, and treating their reciprocals as in AR. PROG.

Ex. 1. Continue to 3 terms each way the series 2, 3, 6. Since $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ are in AR. PROG. with common difference $-\frac{1}{6}$, the AR. series continued each way is 1, $\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$;

$\therefore$ the HARM. series is 1, $\frac{6}{5}$, $\frac{3}{2}$, 2, 3, 6, ∞, $-$ 6, $-$ 3,

where the mark ∞ is called *infinity,* and used to denote the value of $\frac{1}{0}$, which is infinitely great.

Ex. 2. Insert 4 HARM. means between 2 and 12.

We must here insert 4 AR. means between $\frac{1}{2}$ and $\frac{1}{12}$, which by the usual method we shall find to be $\frac{5}{12}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$;

hence the HARM. means required are $\frac{12}{5}$, 3, 4, 6.

EXAMPLES.

671. Continue to 3 terms each way the series 2, $\frac{4}{3}$, 1.

672. Continue to 3 terms each way the series $1\frac{1}{2}$, $2\frac{1}{7}$, $3\frac{3}{4}$.

673. Continue to 3 terms each way the series 1, $1\frac{1}{6}$, $1\frac{2}{5}$.

674. Insert 2 HARM. means between 2 and 4.

675. Insert 6 HARM. means between 3 and $\frac{6}{23}$.

676. Find a fourth HARM. proportional to 6, 8, 12.

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141. *If* A, G, H, *represent, respectively, the* AR., GEOM., *and* HARM. *means between* a *and* b, *then*

$$A = \frac{a+b}{2},\ G = \sqrt{ab},\ H = \frac{2ab}{a+b}.$$

(1) Since A is the AR. mean between a and b, we have (131)

$$b - A = A - a; \therefore 2A = a + b, \text{ and } A = \frac{a+b}{2}.$$

(2) Since G is the GEOM. mean, we have (135)

$$\frac{b}{G} = \frac{G}{a}, \therefore G^2 = ab, \text{ and } G = \sqrt{ab}.^*$$

(3) Since H is the HARM. mean, we have (140, 131), $\frac{1}{b} - \frac{1}{H} = \frac{1}{H} - \frac{1}{a}$; $\therefore$ multiplying every term by abH,

$$aH - ab = ab - bH, \text{ or } aH + bH = 2ab;$$

$$\text{whence } (a+b) H = 2ab, \text{ and } H = \frac{2ab}{a+b}.$$

142. *To prove that* G *is the* GEOM. *mean between* A *and* H; *and that* A, G, H, *are in order of magnitude,* A *being greatest.*

Since $A = \frac{a+b}{2}$, and $H = \frac{2ab}{a+b}$, $\therefore AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$; but $G = \sqrt{ab}$, $\therefore G = \sqrt{AH}$, or G is the GEOM. mean between A and H.

Also $A > H$, if $\frac{a+b}{2} > \frac{2ab}{a+b}$, if $a^2 + 2ab + b^2 > 4ab$,

if $a^2 + b^2 > 2ab$, which is the case (56);

$\therefore A > H$, and, of course, $> G$, whose value (being the GEOM. mean between them) lies between those of A and H.

143. *If* a, b, c, *represent three quantities, then*

1. *when* $\frac{a}{a} = \frac{a-b}{b-c}$, a, b, c, *are in* AR. PROG.;

2. *when* $\frac{a}{b} = \frac{a-b}{b-c}$, ... GEOM. PROG.;

3. *when* $\frac{a}{c} = \frac{a-b}{b-c}$, ... HARM. PROG.

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[*] *a* and *b* must have the *same* sign, or $\sqrt{ab}$ would be *impossible.*

(1) Since $\frac{a}{a} = 1$, we have $1 = \frac{a-b}{b-c}$, and $a - b = b - c$, whence a, b, c, are in AR. PROG.

(2) Since $\frac{a}{b} = \frac{a-b}{b-c}$, we have $ab - ac = ab - b^2$;

$\therefore$ $ac = b^2$, and $\frac{a}{b} = \frac{b}{c}$, whence a, b, c, are in GEOM. PROG.

(3) Since $\frac{a}{c} = \frac{a-b}{b-c}$, we have $ab - ac = ac - bc$;

$\therefore$ dividing every term by abc, $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$,

whence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AR. PROG. and therefore a, b, c in HARM. PROG.

EXAMPLES.

677. Find the AR. GEOM. and HARM. means between 2 and $4\frac{1}{2}$.

678. Find the AR. GEOM. and HARM. means between $3\frac{3}{8}$ and $1\frac{1}{2}$.

679. The sum and difference of the AR. and GEOM. means between two numbers are 9 and 1 respectively; find them.

680. The HARM. mean between two numbers is $\frac{16}{25}$ of the AR., and one of the numbers is 4; find the other.

681. The difference of the AR. and HARM. means between two numbers is $1\frac{4}{5}$; find the numbers, one being four times the other.

682. Find two numbers whose difference is 8, and the HARM. mean between them $1\frac{4}{5}$.

CHAPTER X.

RATIO, PROPORTION, AND VARIATION.

144. The *Ratio* of one quantity to another is the relation which the former bears to the latter in respect of magnitude, the comparison being made by considering, not *by how much* the one is greater or less than the other, but *what number of times*, integral or fractional, the first *contains* the second. The ratio, then, of one quantity to another is the quotient obtained by dividing the former by the latter; thus the ratio of 6 to 3 is $\frac{6}{3}$ or 2, of 15 to 40 is $\frac{15}{40}$ or $\frac{3}{8}$, of $4a$ to $6b$ is $\dfrac{4a}{6b}$ or $\dfrac{2a}{3b}$.

145. The ratio of one quantity to another is expressed by two points placed between them, as $a : b$; and the former is called the *antecedent* term of the ratio, the latter the *consequent*.

A ratio is said to be a ratio of *greater* or *less* inequality, as the antecedent is greater or less than the consequent.

The ratio of $a^2 : b^2$ is called the *duplicate* (i.e. squared) ratio of $a : b$, $a^3 : b^3$ the triplicate ratio of $a : b$, &c.

146. Problems upon ratios are solved by writing them and treating them as ordinary fractions.

Thus ratios are *compared* with one another, by reducing the fractions which represent them to common den$^{\text{rs}}$, and comparing the num$^{\text{rs}}$; and, if these fractions be multiplied together, the resulting fraction is said to be the *ratio compounded of the ratios* represented by them.

Ex. 1. Compare the ratios $5 : 7$ and $4 : 9$.

Here $\frac{5}{7}, \frac{4}{9}$ expressed with common den^{rs} are $\frac{45}{63}, \frac{28}{63}$; whence we see their relative magnitudes, and that $5 : 7 > 4 : 9$.

Ex. 2. Find the ratio of $\frac{5}{7} : \frac{4}{9}$.

This (144) is $\frac{5}{7} \div \frac{4}{9} = \frac{5}{7} \times \frac{9}{4} = \frac{45}{28}$.

Ex. 3. What is the ratio compounded of $2 : 3$, $6 : 7$, $14 : 15$? *Ans.* $\frac{2}{3} \times \frac{6}{7} \times \frac{14}{15} = \frac{8}{15}$ or $8 : 15$.

147. A ratio of less inequality is increased, and of greater inequality diminished, by *adding* the same quantity to both its terms; and, *vice versâ*, by *subtracting* from them.

For $\frac{a+x}{b+x} \gtrless \frac{a}{b}$, as $ab + bx \gtrless ab + ax$, as $bx \gtrless ax$, as $b \gtrless a$;

i.e. as $\frac{a}{b}$ is a ratio of less or greater inequality.

So $\frac{a-x}{b-x} \gtrless \frac{a}{b}$, as $ab - bx \gtrless ab - ax$, as $ax \gtrless bx$, as $a \gtrless b$;

i.e. as $\frac{a}{b}$ is a ratio of greater or less inequality.

EXAMPLES.

683. Compare the ratios $3 : 4$ and $4 : 5$; $13 : 14$ and $23 : 24$; $3 : 7$, $7 : 11$, and $11 : 15$.

684. Of $a + b : a - b$ and $a^2 + b^2 : a^2 - b^2$, which is $>$, supposing $a > b$?

685. Which is $<$ of the ratios $\frac{x+y}{y}$ and $\frac{4x}{x+y}$?

686. Which is $>$ of the ratios $\frac{x^2+y^2}{x+y}$ and $\frac{x^3+y^3}{x^2+y^2}$?

687. Which is $<$ of $x^2 + y^2$ and $\frac{x^6+y^6}{x^4 - x^3y + x^2y^2 - xy^3 + y^4}$?

688. Find the ratio compounded of $3 : 5$, $10 : 21$, and $14 : 15$; of $7 : 9$, $102 : 105$, and $15 : 17$.

689. Find the ratio compounded of

$$\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \text{ and } \frac{a^2-ax+x^2}{a+x}.$$

690. Compound the ratios $\frac{x^2-9x+20}{x^2-6x}$ and $\frac{x^2-13x+42}{x^2-5x}$.

691. Find the ratio compounded of $a+b : a-b$, $a^2+b^2 : (a+b)^2$, and $(a^2-b^2)^2 : a^4-b^4$.

692. What is the ratio compounded of the duplicate ratio of $a+b : a-b$, and the difference of the duplicate ratios of $a : a$ and $a : b$, supposing $a > b$?

693. What quantity must be added to each term of the ratio $a : b$, that it may be equal to the ratio $c : d$?

694. Shew that $a-b : a+b \gtrless a^2-b^2 : a^2+b^2$, as $a : b$ is a ratio of less or greater inequality.

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148. When two ratios are *equal*, the four quantities composing them are said to be *proportional* to one another: thus, if $a : b = c : d$, *i. e.* if $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d, are proportionals. This is expressed by saying that *a is to b as c is to d*, and denoted thus, $a : b :: c : d$.

The first and last quantities in a proportion are called the *Extremes*, the other two the *Means*.

Problems on proportions, like those on ratios, are solved by the use of fractions.

149. *When four quantities are proportionals, the product of the extremes = the product of the means.*

For if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Hence, if three terms of a proportion are given, we can find the other; thus

$$a = \frac{bc}{d},\ b = \frac{ad}{c},\ c = \frac{ad}{b},\ d = \frac{bc}{a}.$$

COR. If $a : b :: b : c$, then $ac = b^2$.

150. *If the product of two quantities = that of other two, the four are proportionals, those of one product being the extremes, and of the other the means.*

For if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$;

and $\therefore a : b :: c : d$, or $a : c :: b : d$, in which proportions a, d are the extremes, and b, c the means.

So if $ac = b^2$, $a : b :: b : c$.

151. If 3 quantities are proportionals, the first has to the third the duplicate ratio of that which it has to the second.

For if $\frac{a}{b} = \frac{b}{c}$, then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$;

$\therefore a : c$ is the duplicate ratio of $a : b$ (145).

152. If 4 quantities form a proportion, they may be arranged so as to form many other proportions, all equally true.

Thus, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$, and $b : a :: d : c$,...(1)

or the four are proportionals when taken *inversely*.

Again, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{c} = \frac{b}{d}$, and $a : c :: b : d$, . . (2)

or the four are proportionals when taken *alternately*.

$$\left.\begin{array}{l} \text{Again, since } \frac{a}{b} = \frac{c}{d},\ \therefore \frac{ma}{mb} = \frac{c}{d}, \text{ or } ma : mb :: c : d \\ \text{similarly} \\ ma : b :: mc : d,\quad a : mb :: c : md,\quad a : b :: mc : md \\ \text{and, in like manner, } \frac{a}{m} : \frac{b}{m} :: c : d, \text{ \&c.} \end{array}\right\} (3)$$

or either the *first* or *fourth* terms may be multiplied or divided by any quantity, if either the *second* or *third* be multiplied or divided by the same.

Hence we may get rid of fractions, when occurring in proportions, by multiplying the 1st and 2nd, or 1st and 3rd, &c. terms by the L. C. M. of their denrs; thus, if

$$\frac{a}{9} : \frac{b}{12} :: \frac{3}{40} : \frac{2}{25},$$

(multiplying 1st and 2nd by 36, 3rd and 4th by 200), we have $4a : 3b :: 15 : 16.$

Again, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$, $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

hence $\frac{a+b}{b} = \frac{c+d}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$, and (by divn) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$;

$$\therefore a+b : b :: c+d : d \ (4), \quad a-b : b :: c-d : d \ (5),$$

$$\text{and } a+b : a-b :: c+d : c-d \ (6);$$

and many other such results may be obtained.

153. *When four magnitudes are proportionals, if any equimultiples whatever be taken of the first and third, and any whatever of the second and fourth, then as the multiple of the first is* >, =, < *that of the second, so is the multiple of the third* >, =, < *that of the fourth.*

Euc. v. Def. 5.

This is the *geometrical* Def. of proportional quantities; for we cannot use the *algebraic* in comparing geometrical magnitudes, lines, areas, solids, &c., since we cannot generally (as with numbers and algc quantities) find how often one such magnitude contains another, nor therefore (144) estimate the two *ratios*, which in the algc Def. are said to be equal. The one, however, results easily from the other.

For if $\frac{a}{b} = \frac{c}{d}$, we have $\left(\text{mult. each by } \frac{m}{n}\right) \frac{ma}{nb} = \frac{mc}{nd}$, where m and n may be any quantities whatever; and hence, if $ma >, =, < nb$, so also is $mc >, =, < nd$.

COR. Hence, if $\frac{a}{b} = \frac{c}{d}$, we have $\frac{ma}{nb} = \frac{mc}{nd}$,

$$\text{and } \therefore \text{ (152. 6) } \frac{ma + nb}{ma - nb} = \frac{mc + nd}{mc - nd}.$$

154. Conversely, *If there be four magnitudes such, that, any equimultiples whatever of the first and third being taken, and any whatever of the second and fourth, if that of the first be >, =, < that of the second, that of the third is also found >, =, < that of the fourth; then these four quantities are proportionals.* Euc. v. Def. 5.

For, let a, b, c, d be such that, any equimultiples, ma, mc, being taken of the first and third, and any nb, nd, of the second and fourth, when $ma >, =, < nb$, then also $mc >, =, < nd$; and let e be the fourth proportional to a, b, c.

Then, since $\frac{a}{b} = \frac{c}{e}$, $\therefore \frac{ma}{nb} = \frac{mc}{ne}$ for all values of m and n; take m and n such that $ma = nb$, then also $mc = ne$: but when $ma = nb$, by our hyp., $mc = nd$; hence $nd = ne$, or $d = e$; and $\therefore \frac{a}{b} = \frac{c}{d}$, or a, b, c, d are proportionals.

155. If $a : b :: c : d$, and $b : e :: d : f$, then $a : e :: c : f$.

For $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{c} = \frac{b}{d}$; and $\frac{b}{e} = \frac{d}{f}$, $\therefore \frac{b}{d} = \frac{e}{f}$;

hence $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$, and $\frac{a}{e} = \frac{c}{f}$ or $a : e :: c : f$.

This is the proposition *ex æquali*, referred to in Euc. VI.

156. If $a : b :: c : d$, and $e : f :: g : h$, then $ae : bf :: cg : dh$.

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{e}{f} = \frac{g}{h}; \quad \therefore \frac{ae}{bf} = \frac{cg}{dh}.$$

This is called *compounding* the proportions $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{g}{h}$: and so with any number of such proportions.

157. If $a : b :: c : d :: e : f ::$ &c.

then $a : b :: a + c + e +$ &c. $: b + d + f +$ &c.

$$\text{For since } \frac{a}{b} = \frac{a}{b}, \ \frac{a}{b} = \frac{c}{d}, \ \frac{a}{b} = \frac{e}{f}, \text{ \&c.}$$

$$\therefore ab = ba, \quad ad = bc, \quad af = be, \quad \text{\&c.};$$

$$\therefore ab + ad + af + \text{\&c.} = ba + bc + be + \text{\&c.},$$

$$\text{or } a(b + d + f + \text{\&c.}) = b(a + c + e + \text{\&c.});$$

$$\therefore (150)\ a : b :: a + c + e + \text{\&c.} : b + d + f + \text{\&c.}:$$

i.e. if several quantities be proportionals, as one of the antecedents is to its consequent so is the sum of all the antecedents to the sum of all the consequents.

Ex. 1. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

$$\text{Since } d = \frac{bc}{a}, \ (149) \text{ this is } \frac{\frac{1}{3} \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{6}.$$

Ex. 2. If $a : b :: c : d :: e : f$, shew that

$$a : b :: ma + nc + pe : mb + nd + pf.$$

Since $a : b :: c : d :: e : f$,

$\therefore ma : mb :: nc : nd :: pe : pf$, (152)

$\therefore ma : mb :: ma + nc + pe : mb + nd + pf$, (157)

or $a : b :: ma + nc + pe : mb + nd + pf$.

EXAMPLES.

695. Find a fourth proportional to 3, 5, 6; to 12, 5, 10; to $\frac{2}{7}$, $\frac{3}{4}$, $\frac{5}{6}$.

696. Find a third proportional to 4, 6; to 2, 3; to $\frac{5}{3}$, $\frac{5}{7}$.

697. Find a mean proportional to 4, 9; to 4, $\frac{16}{25}$; to $1\frac{7}{9}$, $1\frac{9}{16}$.

698. If $a : b :: c : d$, then $a - b : a :: c - d : c$.

699. If $a : b :: c : d$, then $ma + nb : b :: mc + nd : d$, and $ma + nb : mc + nd :: pa + qb : pc + qd$.

700. If $a : b :: c : d$, and $m : n :: p : q$, then $ma + nb ; ma - nb :: pc + qd : pc - qd$.

701. If $a : b :: b : c$, then $a^2 - b^2 : a :: b^2 - c^2 : c$

702. If $a : b :: c : d :: e : f$, then $a - e : b - f :: c : d$.

703. If $a : b :: c : d :: e : f$, then
$ma + nc + pe : mb + nd + pf :: ma - nc + pe : mb - nd + pf$.

704. If $\frac{a}{b} = \frac{c}{d}$, then $(a + d) - (b + c) = \frac{(a - b)(a - c)}{a}$.

705. If $a : b :: b : c$, then $a - 2b + c = \frac{(a - b)^2}{a} = \frac{(b - c)^2}{c}$.

706. If $\frac{a}{b} = \frac{c}{d}$, then $\left(\frac{1}{a} + \frac{1}{d}\right) - \left(\frac{1}{b} + \frac{1}{c}\right) = \frac{(a-b)(a-c)}{abc}$.

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158. Since the values of any two quantities will depend on the values we give to the letters they contain, the value of their ratio will generally change with any change in the values of the letters: thus,

$$\frac{x^2 + 3x}{2x + 1} = \frac{4}{3}, \text{ if } x = 1, \text{ or } = \frac{10}{5}, \text{ if } x = 2, \text{ \&c.}$$

But it *may* happen, that the two quantities are such, that their ratio will be *constant*, whatever be the values we give to the letters they contain; thus, $\frac{x^2 + 3x}{2x^2 + 6x} = \frac{1}{2}$, for all values of x. In such a case, one quantity is said to *vary as* the other.

The sign used to denote variation is $\propto$ (read *varies as*), thus, $x^2 + 3x \propto 2x^2 + 6x$.

159. *If* A $\propto$ B, *then* A = mB, *where* m *is some constant quantity.*

For if $A \propto B$, then (158) $\frac{A}{B}$ is constant, whatever be the separate values of the quantities represented by A and B, arising from the values given to the letters they contain; putting then m to denote this constant quantity, (which may be integral or fractional), we have $\frac{A}{B} = m$, or $A = mB$.

Cor. Hence one quantity varies as another, when they are connected by some fixed multiplier, integral, or fractional.

160. *If* A $\propto$ B, *and* a, b *be certain values of* A *and* B, *corresponding to given values of the letters they contain, then whatever other values* A *and* B *may have, we shall have always* A : a :: B : b.

For since $A \propto B$, the ratio of $A : B$ will always be the same, whatever be their separate values (158);

$$\text{hence } \frac{A}{B} = m = \frac{a}{b}, \text{ and } \frac{A}{a} = \frac{B}{b}; \quad \therefore A : a :: B : b.$$

Cor. Hence one quantity varies as another; when, if their values are both changed to other values, the four are proportionals.

161. 1. If $A \propto B$, then $A = mB$, and is said to vary *directly* as B;

2. If $A \propto \frac{1}{B}$, then $A = \frac{m}{B}$, and is said to vary *inversely* as B;

3. If $A \propto BC$, then $A = mBC$, and is said to vary *jointly* as B and C;

4. If $A \propto \frac{B}{C}$, then $A = m\frac{B}{C}$, and is said to vary *directly* as B, and *inversely* as C.

162. (1) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For let $A=mB$, $B=nC$; then $A=mnC$,* and $\therefore A \propto C$.

So also, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

(2) If $A \propto C$ and $B \propto C$,

then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

For let $A = mC$, $B = nC$;

then $A \pm B = mC \pm nC = (m \pm n)C$, and $\therefore A \pm B \propto C$;

and $\sqrt{AB} = \sqrt{mC \times nC} = \sqrt{mnC^2} = \sqrt{mn}\,C$; and $\therefore \sqrt{AB} \propto C$.

(3) If $A \propto BC$, then $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$.

For let $A = mBC$, then $B = \frac{1}{m} \cdot \frac{A}{C}$,

and $\therefore B \propto \frac{A}{C}$; and so $C \propto \frac{A}{B}$.

(4) If $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For let $A = mB$, $C = nD$; then $AC = mnBD$, and $\therefore AC \propto BD$.

163. *If* A *so depend on* B *and* C, *that, when* C *is kept constant,* A $\propto$ B, *and when* B *is kept constant,* A $\propto$ C; *then, generally,* i.e. *when neither is kept constant,* A $\propto$ BC.

For since $A \propto B$, when C is constant, we have $A = mB$, where m is some *constant,* and *may,* therefore, contain the constant C, but not B. Again, since $A \propto C$, when B is constant, we have $A = nC$, where n is some *constant,* and *may,* therefore, contain the constant B, but not C. Comparing this result with the former, we see that m must be of the form pC, and n of the form pB, where p is a constant, containing neither B nor C; and $\therefore A = pBC$, or $A \propto BC$, when all three are changeable.

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* Since m, n, are constant, so also are mn, $m \pm n$, $\sqrt{mn}$, &c.

The above result may similarly be proved for any No. of quantities, B, C, D, &c.; *i. e.* if any quantity vary separately as each of several others, when the rest are constant, it varies as their product, when all are changeable.

164. Single and Double Rule of Three sums are solved upon the principles of Variation and Proportion.

In *Single* Rule of Three, we have given corresponding values of two things which vary as one another, and have then to find the change produced in one of them, when a given change is made in the other.

Thus is the question, 'If 57 cwt. cost £216, what will 95 cwt. cost?' we have given 57 cwt. and £216, corresponding quantities of *weight* and *price*; and have to find to what £216 will be changed, when 57 cwt. is changed to 95 cwt. Now weight $\propto$ price *directly;*

$$\therefore 57 : 95 :: 216 : \text{the answer} = \frac{95 \times 216}{57} = 360 \ldots (149).$$

Again, in the question, 'How many men would do in 168 days a piece of work which 108 men can do in 266 days?' we have given 266 days and 108 men, corresponding quantities of *time* and *labour*; and have to find to what 108 men must be changed, when 266 days are changed to 168 days. Now the time for a given work is greater as the No. of men employed is less, and *vice versâ*, *i. e.* time $\propto$ No. of men *inversely;*

$$\therefore \tfrac{1}{266} : \tfrac{1}{168} :: 108 : \text{the answer} = \tfrac{1}{168} \times 108 \div \tfrac{1}{266} = 171.$$

From (152. 3) we see that we may simplify a Rule of Three sum by dividing the 1st and 2nd, or 1st and 3rd terms, by any common factor which they may contain.

165. In *Double* Rule of Three, we have given one thing, which varies separately as *each* of several others, if the rest be constant, and, therefore, varies as their product, when all are changeable. (163). Having given then one value of the former or single quantity, and corresponding values of the others, we have to find to what the first will be changed, when the latter are changed to others also given.

Thus, in the question, 'If 12 horses plough 11 acres in 5 days, how many horses would plough 33 acres in 18 days?' the single term is *horses*, and the others are *acres* and *days*. Now if the No. of days were constant, the No. of horses would vary as that of acres *directly;* again, if the No. of acres were constant, the No. of horses would vary as that of days *inversely*; hence the No. of horses $\propto \dfrac{\text{No. of acres}}{\text{No. of days}}$, when both are changeable;

$$\therefore \tfrac{11}{5} : \tfrac{33}{18} :: 12 : \text{the answer} = \tfrac{33}{18} \times 12 \div \tfrac{11}{5} = 10.$$

Ex. 1. If $a \propto b^2c$, and 1, 2, 3, be contemporaneous values of a, b, c, express a in terms of b and c.

Since $a \propto b^2c$, $\therefore a = mb^2c$, where we have to find m; now, by the question, when $b = 2$ and $c = 3$, a becomes 1;

$$\therefore 1 = m \times 2^2 \times 3 = 12m, \text{ or } m = \frac{1}{12}, \text{ and } \therefore a = \frac{b^2c}{12}.$$

Ex. 2. If $y =$ the sum of two quantities, one of which $\propto x$ and the other $\propto x^2$, and when $x = 1$, $y = 6$, when $x = 2$, $y = 20$; express y in terms of x.

Here $y = mx + nx^2$, where we have to find m and n;

$$\left.\begin{array}{l} \text{but when } x = 1,\ y = 6,\ \therefore\ 6 = m + n \\ \text{and when } x = 2,\ y = 20,\ \therefore\ 20 = 2m + 4n \end{array}\right\},$$

from which equations $m = 2$, $n = 4$, and $\therefore y = 2x + 4x^2$.

EXAMPLES.

707. If $A \propto B$, then $A^2 \pm B^2 \propto AB$,
and $A^2B \pm AB^2 \propto (A \pm B)^3$.

708. If $A + B \propto C$, and $B + C \propto A$,
then $A + C \propto B$, and $A^2 \propto BC$.

709. If $A \pm B \propto B \pm C$, then $A \propto B \propto C$.

710. If $A \propto B$, $AC \propto D^2$, and $B^2 \propto CD$,
then $(A + B + C)^4 \propto ABCD \propto A^4 + B^4 + C^4$.

711. If $xy \propto x^2 + y^2$, and 3, 4, be contemporaneous values of x and y, express xy in terms of $x^2 + y^2$.

712. If $y =$ the sum of two quantities, whereof one is constant and the other $\propto x$ *inversely*, and when $x = 2$, $y = 0$, when $x = 3$, $y = 1$; find the value of y, when $x = 6$.

713. If $y =$ the sum of two quantities, whereof one is constant, and the other $\propto xy$, and when $x = 2$, $y = -2\frac{1}{3}$, when $x = -2$, $y = 1$; express y in terms of x.

714. If $y =$ the sum of three quantities, which vary as x, x^2, x^3 respectively, and when $x = 1, 2, 3$, $y = 6, 22, 54$ respectively, express y in terms of x.

715. If $y =$ the sum of three quantities, of which the first $\propto x^2$, the second $\propto x$, and the third is constant; and when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively, express y in terms of x.

716. The area of any triangle varies jointly as any side, and the perpendicular let fall upon it from the opposite angle; express the area of the right-angled triangle ABC in terms of the sides AC, BC, containing the right angle, it being found that, when the sum of the two sides is 14 feet and the hypothenuse 10 feet, the area is 24 square feet.

CHAPTER XI.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

166. The *Variations* of any No. of quantities are the different *arrangements* which can be made of them, taking a certain No. at a time together; thus the Var^{ns} of a, b, c taken *two* together are ab, ba, ac, ca, bc, cb.

When *all* the given quantities are taken together, the Var^{ns} are called *Permutations.*

167. *The No. of Var^{ns} of* n *different things, taken* r *together, is* $n(n-1)(n-2)\ldots\ldots(n-r+1)$.

Let there be n different things, a, b, c, d, &c.

The No. of Var^{ns} which can be formed of these n things, taken *singly*, is, of course, n.

Now let us remove a; there will then be $n-1$ things, b, c, d, &c., and the Var^{ns} of these taken *singly*, will (as before) be $n-1$; if then we set a before each of these, there will be $n-1$ Var^{ns} of n things, a, b, c, d, &c. taken *two* and *two* together, in which a stands first; similarly there will be $n-1$ such Var^{ns}, in which b stands first; and so of the rest: therefore, on the whole there will be $n(n-1)$ Var^{ns} of n things taken *two* and *two* together.

Let us again remove a; there will be $n-1$ things, b, c, d, &c., and the Var^{ns} of these, taken *two* and *two* together, will be $(n-1)(n-2)$ by what precedes; and, by the same course of reasoning, it will appear that, on the whole, there will be $n(n-1)(n-2)$ Var^{ns} of n things taken *three* and *three* together.

Suppose then this law to hold for the No. of Var^{ns} of n things a, b, c, d, &c. taken $r - 1$ together, which would be, therefore,

$$n(n-1)(n-2) \dots \{n-(r-1)+1\},$$
$$\text{or } n(n-1)(n-2) \dots (n-r+2):$$

now let a be removed; there will then be $n - 1$ things b, c, d, &c., and the Var^{ns} of these, taken $r - 1$ together, would be found from the preceding result, by writing in it $n - 1$ for n, and would, therefore, be

$$(n-1)(n-2) \dots\dots (n-r+1):$$

if now we set a before each of these, there would be $(n-1)(n-2) \dots (n-r+1)$ Var^{ns} of n things a, b, c, d, &c. taken r together, in which a stands first; similarly, when b stands first, and so of the rest: therefore, on the whole, there would be $n(n-1)(n-2) \dots (n-r+1)$ Var^{ns} of n things taken r together.

If then the formula represent correctly the No. of Var^{ns} of n things when taken $r - 1$ together, it would also when taken r together: but we have shewn it to be true when they are taken 1, 2, or 3 together; therefore when taken 4 together; and, therefore, when 5 together, &c.: it is therefore generally true for all values we can give to r.

168. Hence denoting by V_1, V_2, V_3, &c. V_r the No. of Var^{ns} of n things taken 1, 2, 3, &c. r together, we have, from the preceding formula,

$$V_1 = n,\ V_2 = n(n-1),\ V_3 = n(n-1)(n-2), \text{ \&c.}$$
$$V_r = n(n-1) \dots (n-r+1).$$

Cor. If $r = n$, or *all* the quantities are taken together, then the No. of Perm^{ns} (P) of n things, is

$$n(n-1)(n-2) \dots (n-n+1) = n(n-1)(n-2) \dots 1;$$

or, reversing the order of the factors,

$$P = 1.2.3 \ldots\ldots n.$$

169.' *The No. of Perm*^{ns} *of* n *letters, whereof* p *are* a*'s*, q *are* b*'s*, r *are* c*'s*, *&c. is*

$$\frac{1.2.3 \ldots \mathrm{n}}{1.2.3 \ldots \mathrm{p} \times 1.2.3 \ldots \mathrm{q} \times \&c.}.$$

For let N be the No. of such Perm^{ns}. Suppose now that in any *one* of them we change the p a's into *different* letters; then these letters might be arranged (168, Cor.) in $1.2.3 \ldots p$ different ways, and so instead of this *one* Perm^n, in which p letters would have been a's, we shall now have $1.2.3 \ldots p$ *different* Perm^{ns}. The same would be true for *each* of the N Perm^{ns}; hence if the p a's were changed to *different* letters, we should have altogether $1.2.3 \ldots p \times N$ different Perm^{ns} of n letters, whereof still q are b's, r are c's, &c.

Now if in these the q b's were changed to different letters, we should similarly have

$$1.2.3 \ldots q \times 1.2.3 \ldots p \times N$$

different Perm^{ns} of n things, whereof still r would be c's, &c.; and so on until *all* the n letters are different ones; but when this is the case we know (168, Cor.) that their whole number of permutations = $1.2.3 \ldots n$; hence

$$1.2.3 \ldots p \times 1.2.3 \ldots q \times \&c. \times N = 1.2.3 \ldots n,$$

$$\text{and } N = \frac{1.2.3 \ldots n}{1.2.3 \ldots p \times 1.2.3 \ldots q \times \&c.}.$$

Ex. 1. How many changes can be rung with 6 bells out of 8? How many with the whole peal?

Here $V_6 = 8.\ 7.\ 6.\ 5.\ 4.\ 3 = 20160$,

and $P = 8.\ 7.\ 6.\ 5.\ 4.\ 3.\ 2.\ 1 = 40320$.

Ex. 2. How many *different* words may be made with all the letters of the expression a^3b^2c?

Here there are 6 letters, whereof 3 are *a*'s, and 2 *b*'s;

$$\therefore N = \frac{1.\,2.\,3.\,4.\,5.\,6}{1.\,2.\,3 \times 1.\,2} = 60.$$

Ex. 3. What No. of things is that, whereof the No. of Var^{ns} taken 3 together, is 20 times as great as the No. of Var^{ns} of half the same No. of things taken 2 together?

Here, if *n* denote the No. of things required, we have

$$n(n-1)(n-2) = 20\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right), \text{ whence } n = 6.$$

EXAMPLES.

717. How many changes may be rung with 5 bells out of 6, and how many with the whole peal?

718. In how many different ways may 7 persons seat themselves at table?

719. How many different words may be made of all the letters of the words *division* and *insincere?*

720. How many different words may be made of the letters of the expression $a^4b^3c^2d$?

721. The No. of Var^{ns}, 3 together: the No., 4 together :: 1 : 6; find the No. of things.

722. How many different words may be made of all the letters of the words *Oroonoko* and *Mississippi?*

723. The No. of things: the No. of Var^{ns}, 3 together :: 1 : 20; find the No. of things.

724. The No. of Var^{ns} of *n* things, 3 together : the No. of Var^{ns} of $n + 2$ things, 3 together :: 5 : 12; find *n*.

725. The No. of Var^{ns} of *n* things, 4 together : the No. of Var^{ns} of $\frac{2n}{3}$ things, 4 together :: 13 : 2; find *n*.

726. If the No. of Var^{ns} of n things, 3 together, be 12 times as great as the No. of Var^{ns} of $\frac{n}{2}$ things, 3 together, what is the No. of Perm^{ns} of the same n things?

727. Of what number of things are the Perm^{ns} 720?

728. There are 7 letters, of which a certain No. are *a*'s; and 210 different words can be made of them; how many *a*'s are there?

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170. The *Combinations* of any No. of quantities are the different sets that can be made of them, taking a certain No. together, without regard to the order in which they are placed; thus, the Comb^{ns} of *a, b, c, d,* taken 3 together, are *abc, abd, acd, bcd.*

It is readily seen that each *Comb^{n}* will supply as many corresponding *Var^{ns}*, as the No. of quantities it contains admits of Perm^{ns}; thus, the Comb^{n} *abc* supplies the 1.2.3 or 6 Var^{ns} *abc, acb, bac, bca, cab, cba.*

171. *The No. of Comb^{ns} of* n *different things, taken* r *together, is*

$$\frac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots.r}.$$

For (170) each *Comb^{n}* of r things will supply $1.2.3..r$ *Var^{ns}* of r things; hence if C_r denote the whole No. of Comb^{ns} of n things, r together, we have

$$1.2.3\ldots.r \times C_r = \text{whole No. of Var}^{ns},\ r \text{ together},$$
$$= V_r = n(n-1)(n-2)\ldots.(n-r+1);$$
$$\therefore C_r = \frac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots.r}.$$

172. Hence

$$C_1 = \frac{n}{1},\ C_2 = \frac{n(n-1)}{1.2},\ C_3 = \frac{n(n-1)(n-2)}{1.2.3},\ \&c.$$

where it is plain that C_1, C_2, C_3, &c. have the same form as the coeff^{s} of the binomial $(1+x)^n$.

COR. The sum of all the Comb^{ns} that can be made of n things, taken 1, 2, 3, &c. n together

$$= C_1 + C_2 + \&c. + C_n = 2^n - 1. \quad (130.)$$

173. As in (123) it may be shewn that $C_r = C_{n-r}$, or the No. of Comb^{ns} of n things, taken r together = the No. taken $n - r$ together. The Comb^{ns} of one of these sets are said to be *Supplementary* to those of the other.

Ex. Find the No. of Comb^{ns} of 9 things taken 7 together.

Here, instead of finding C_7, we take $C_2 = \frac{9.8}{1.2} = 36$; and it is plain that $C_7 = \frac{9.8.7.6.5.4.3}{1.2.3.4.5.6.7}$, which (on striking out the common factors) reduces to $\frac{9.8}{1.2} = C_2$.

174. *To find what No.* r *out of* n *things, must be taken together, that the No. of Comb^{ns} so formed may be the greatest possible.*

Since C_1, C_2, &c. increase until we come to the *greatest*, after which (173) they decrease in corresponding order, we must take r the *least* No. which gives

$$C_r > C_{r-1}, \text{ or } \frac{n(n-1)..(n-r+2)(n-r+1)}{1.2 \dots (r-1)r} > \frac{n(n-1)..(n-r+2)}{1.2\dots(r-1)};$$

hence (striking out common factors)

$$\frac{n-r+1}{r} > 1, \text{ and } n-r+1 > r, \text{ or } n+1 > 2r \text{ and } \frac{n+1}{2} > r,$$

$$\text{i.e. } r \text{ is the integer next} < \frac{n+1}{2}.$$

If n be *even*, $r = \frac{n}{2}$: if n be odd, and $\therefore \frac{n+1}{2}$ an integer, $r = \frac{n+1}{2} - 1 = \frac{n-1}{2}$; but, in this case, since (173)

$C_r = C_{n-r}$, the No. taken $\frac{n-1}{2}$ together = the No. taken $n - \frac{n-1}{2}$ or $\frac{n+1}{2}$ together, and $\therefore r$ may be either $\frac{n-1}{2}$ or $\frac{n+1}{2}$.

Ex. If $n = 6$, $r = \frac{n}{2} = 3$, and $C_3 = \frac{6.5.4}{1.2.3} = 20$;

if $n = 7, r = \frac{n \pm 1}{2} = 4$ or 3, and $C_3 = 35 = C_4$.

175. *If there be several sets, containing* n_1, n_2, n_3, *&c. things respectively, the No. of Comb^ns formed by taking one out of each set, is* $n_1 . n_2 . n_3$. *&c.*

For let there be two such sets, as a_1, a_2, &c. b_1, b_2, &c; then, since each of the n_1 things in the first may be combined with each of the n_2 in the second, the No. of Comb^ns of *two* things, one out of each set, thus formed, will be $n_1 . n_2$. Now if there be a third set of n_3 things, c_1, c_2, &c., each of the former $n_1 . n_2$ Comb^ns may be taken with each of these n_3 things, and thus form $n_1 . n_2 . n_3$ Comb^ns of *three* things, one out of each set; and so on.

Cor. If there be the same No., n, in each set, and r such sets, the No. of Comb^ns thus formed is n^r.

Ex. 1. How many Comb^ns may be made of 10 things, taken 3 and 6 together?

Here $C_3 = \frac{10.9.8}{1.2.3} = 120$, and $C_6 = C_4 = \frac{10.9.8.7}{1.2.3.4} = 210$.

Ex. 2. There are 4 bags, containing 3, 4, 5, 6 balls respectively; how many different sets of balls can I draw, one from each bag? how many, also, if they each contain 3 balls? *Ans.* As in (175), $3.4.5.6 = 360$; and $3^4 = 81$.

Ex. 3. The No. of Comb^{ns} of $n+1$ things, 3 together, exceeds by 45 the No. of Comb^{ns} of n things, 3 together; find n.

Here $\frac{(n+1)n(n-1)}{1.2.3} = \frac{n(n-1)(n-2)}{1.2.3} + 45$, whence $n=10$.

EXAMPLES.

729. How many Comb^{ns} can be made of 9 things, 4 together? how many, 6 together?

730. How many Comb^{ns} can be made of 11 things, 4 together? how many, 7 together?

731. A person having 15 friends, on how many days might he invite a different party of 10?

732. How often might a common die be thrown, so as to expose five *different* faces?

733. How many Comb^{ns} can be made in all of 6 things, taken 1, 2, 3, 4, 5, 6 together?

734. Four persons are chosen by lot out of 10; in how many ways can this be done? and how often would any one person be chosen?

735. On how many nights may a *different* guard be posted of 6 men out of 60? on how many of these would any one man be taken?

736. The No. of Comb^{ns} of $\frac{n}{3}$ things, 2 together, is 30; find n.

737. The No. of Comb^{ns} of n things, 3 together, is $\frac{5}{18}$ of the No., 5 together; find n.

738. The No. of Comb^{ns} of $n+1$ things, 4 together, is 9 times the No. of Comb^{ns} of n things, 2 together; find n.

739. The No. of Comb^{ns} of $\frac{n}{2}$ things, 4 together, is $3\frac{3}{4}$ of the No. of Comb^{ns} of $\frac{n}{3}$ things, 3 together; find n.

740. How many words of 6 letters may be made out of the 26 letters of the alphabet, with 2 out of the 5 vowels in every word?

741. Find the greatest No. of Comb^{ns} that can be made out of 10 things; and the whole No. when taken 1, 2, 3, &c. 10 together.

742. How many letters of the word *holidays* should be taken together so as to produce the greatest No. of different words? What is the difference between that No. of words, and the greatest which can be produced by taking letters of the word *universal*?

743. A person wishes to make up as many *different* dinner parties as he can, out of an acquaintance of 24; how many should he invite at a time?

744. Six yachts are to be so arranged in a regatta, that there may be the greatest possible No. of different matches; how many then must sail in each match, and how many matches would there be?

745. In a peal of 9 bells, how many must be sounded so as to ring the greatest possible No. of changes, and what will that No. be?

746. In what numbers should 20 men be combined so as to form the greatest possible No. of different companies? In how many of these would the same man be found?

CHAPTER XII.

NOTATION.

176. *Notation* is the method of expressing numbers by means of a series of powers of some one fixed number, which is said to be the *radix* or *base* of the *scale*, in which the different numbers are expressed.

Thus in common Arithmetic, all Nos. are expressed in a scale whose base is 10; for 3578 denotes $3000 + 500 + 70 + 8$, *i. e.* $3.10^3 + 5.10^2 + 7.10 + 8$: so also 376, when expressed in a scale whose radix is 12, is $2.12^2 + 7.12 + 4$, which $= 288 + 84 + 4 = 376$.

177. *If* r *be any integer, any No.* N *may be expressed by a series of powers of* r *in the form*

$$\mathrm{N} = \mathrm{a}_n \mathrm{r}^n + \mathrm{a}_{n-1} \mathrm{r}^{n-1} + \&c. + \mathrm{a}_2 \mathrm{r}^2 + \mathrm{a}_1 \mathrm{r} + \mathrm{a}_0,$$

where the coeff^{s} a_n, a_{n-1}, &c. *are integers all less than* r.

For divide N by the greatest power of r it contains, suppose r^n; and let the quotient be a_n (which will, of course, be $< r$), and the remainder N_1: then $N = a_n r^n + N_1$.

Similarly $N_1 = a_{n-1} r^{n-1} + N_2$, $N_2 = a_{n-2} r^{n-2} + N_3$, &c. and thus continuing the process until the remainder becomes $< r$, a_0 suppose, we have

$$N = a_n r^n + a_{n-1} r^{n-1} + \&c.\ a_2 r^2 + a_1 r + a_0.$$

Some of the coeff^{s} a_0, a_1, a_2, &c. may vanish, but none can be $> r$. Their values then may range from 0 to $r - 1$, and these different values, which they may assume for any value of r, are called the *digits* of the corresponding scale.

Including *zero*, therefore, there will be r digits in the scale of r; thus in the scale of 12, the digits will be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t* and *e*, where *t* and *e* are used to denote the digits 10 and 11.

178. *The greatest No. that can be expressed with* n *digits in the scale of* r *is* $r^n - 1$, *the least is* r^{n-1}.

For let $N = a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \&c. + a_2 r^2 + a_1 r + a_0$: then N will be *greatest*, when a_{n-1}, a_{n-2}, &c. have each of them their greatest possible value, i.e. $r - 1$; so that

$$N = (r - 1)(r^{n-1} + r^{n-2} + \&c.\ r^2 + r + 1)$$

$$= (r - 1)\frac{r^n - 1}{r - 1}\ (137) = r^n - 1;$$

and N will be *least*, when $a_{n-1} = 1$, and a_{n-2}, a_{n-3}, &c. all vanish, so that $N = r^{n-1}$.

Ex. What are the greatest and least Nos. that can be expressed in the scale of 7 with 7 digits?

These are $7^7 - 1$ and 7^6, or 823542 and 117649: and it is plain that the greatest and least Nos. that can be expressed with 7 digits in the scale of 7 (where the highest digit is 6) are 6666666 and 1000000, which, reduced to the common scale, will produce the above Nos.

179. In the *Binary* scale, the radix is 2; the *Ternary*, 3; the *Quaternary*, 4; the *Quinary*, 5; the *Senary*, 6; &c.; the *Denary* or *Decimal*, 10; the *Undenary*, 11; the *Duodenary*, or *Duodecimal*, 12; &c.

All Nos. are supposed to be expressed in the common or denary scale, unless the contrary is mentioned.

180. *Any No. in the scale of* r, *and the sum of its digits, when divided by* $r - 1$, *leave the same remainders.*

For let $N = a_n r^n + \&c. + a_2 r^2 + a_1 r + a_0$,

$$\text{or } = a_0 + a_1 r \qquad + a_2 r^2 + \&c. + a_n r^n;$$

$$\text{then } N = a_0 + a_1 \qquad + a_2 \qquad + a_3 + \&c.$$

$$+ a_1(r - 1) + a_2(r^2 - 1) + a_3(r^3 - 1) + \&c.;$$

where in the lower line all the factors $r-1$, r^2-1, r^3-1, &c. are divisible by $r-1$, and therefore, $N \div (r-1)$ will leave the same rem^{r} as the upper line $a_0 + a_1 + a_2 +$ &c., or the sum of its digits, similarly divided.

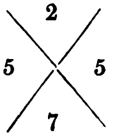
Cor. 1. Any No. in the scale of r is exactly divisible by $r-1$, if the sum of its digits be so divisible.

Cor. 2. Hence in the common scale, any No. and the *sum of its figures* leave the same rem^{r}, when divided by $10-1$ or 9; and the No. itself is divisible by 9, if the sum of its figures be so divisible.

The same properties hold also for 3; since the lower line, in the above, being divisible by 9, will be also by 3; and therefore if the upper line, *i.e.* the sum of the digits, be divided by 3, it will leave the same rem^{r} as $N \div 3$.

Cor. 3. Hence we may prove the common process of *casting out nines*, in order to test the truth of a sum in Multiplication.

$$\begin{array}{lr} \text{Mult.} & 875 \\ \text{by} & 358 \\ \hline & 7000 \\ & 4375 \\ & 2625 \\ \hline & 313250 \end{array}$$



Drawing a cross, as above, we set in two opposite angles the rem^{rs} found by *adding up the figures* of each of the two given Nos., and dividing the results by 9. These are 2 and 7, which we multiply, and divide the result by 9, and place the rem^{r} 5 in a third angle of the cross. Then, if the sum be worked correctly, and we divide the final product, or the *sum of its figures*, by 9, we shall find the rem^{r} to be also 5, which we place in the fourth angle, and the test is completed.

To prove this, let P and Q be the two Nos.; and let $P = 9p + a$, $Q = 9q + \beta$, where a, β, being the rem^{rs} upon dividing P, Q, respectively by 9, will be the same (Cor. 1) as these found by *adding the figures* of P, Q, and dividing the results by 9. They will therefore be the Nos. first set in the two opposite angles of the cross.

Now $PQ = 81pq + 9aq + 9\beta p + a\beta$; and, therefore, the rem^{r} upon dividing PQ, the *product* of P and Q, or the *sum of the figures* in PQ, by 9, will be the same as that left on dividing $a\beta$ by 9.

N.B. This method only shews when we are *wrong*, but not always that we are *right*, in our sum; for if we have omitted a 9, or any multiple of 9, or misplaced figures, &c.; these errors would not be detected by this process.

181. *Any No. in the scale of* r, *and the difference of its odd and even digits, when divided by* r + 1, *leave the same remainder.*

$$\text{For } N = a_0 + a_1 r + a_2 r^2 + \&c. + a_n r^n;$$
$$\therefore N = a_0 - a_1 + a_2 - a_3 + \&c.$$
$$+ a_1 (r + 1) + a_2 (r^2 - 1) + a_3 (r^3 + 1) + \&c.;$$

where the lower line is divisible by $r + 1$, and therefore $N \div (r + 1)$ will leave the same rem^{r} as the upper line $a_0 - a_1 + a_2 - a_3 + \&c.$ *i. e.* the difference of the odd and even digits, so divided.

Cor. 1. Any No. in the scale of r is divisible by $r + 1$, if the difference of its odd and even digits be so divisible.

Cor. 2. Hence in the common scale, any No. and the *diff^{ce} of its odd and even figures*, when divided by $10 + 1$ or 11, leave the same rem^{r}; and the No. itself is divisible by 11, if the diff^{ce} of its odd and even figures be so divisible.

182. *To express any proposed No. in a given scale.*

Let N be the given No. which, when expressed in the scale of r, will assume the form

$$N = a_n r^n + a_{n-1} r^{n-1} + \&c. + a_2 r^2 + a_1 r + a_0;$$

we are to shew how the digits a_n, a_{n-1}, &c. may be found.

Divide N by r; then we shall have

$$\frac{N}{r} = a_n r^{n-1} + a_{n-1} r^{n-2} + \&c. + a_2 r + a_1 + \frac{a_0}{r},$$

i. e. an integral quotient, $a_n r^{n-1} + \&c. + a_1 = N_1$, suppose, with rem^r^ a_0; hence the rem^r^, upon dividing N by r, is a_0, the *last* of the digits.

Again, divide N_1 by r; then we shall have

$$\frac{N_1}{r} = a_n r^{n-2} + a_{n-1} r^{n-3} + \&c. + a_2 + \frac{a_1}{r} = N_2 + \frac{a_1}{r};$$

hence the rem^r^, upon dividing N_1 by r_1 is a_1, the *last but one* of the digits; and so dividing N_2 by r, we get a_2, &c.

Ex. Express 3700 in the *quinary*, and 37694 in the *octenary* scale.

$$\begin{array}{rl}
5\,\underline{)\,3700} & \\
5\,\underline{)\,740} & \ldots 0 \\
5\,\underline{)\,148} & \ldots 0 \\
5\,\underline{)\,29} & \ldots 3 \\
5\,\underline{)\,5} & \ldots 4 \\
1 & \ldots 0
\end{array}
\qquad
\begin{array}{rl}
8\,\underline{)\,37694} & \\
8\,\underline{)\,4711} & \ldots 6 \\
8\,\underline{)\,588} & \ldots 7 \\
8\,\underline{)\,73} & \ldots 4 \\
8\,\underline{)\,9} & \ldots 1 \\
1 & \ldots 1
\end{array}$$

Ans. 104300. *Ans.* 111476.

183. The common processes of Arithmetic are carried on with these, as with ordinary Nos., observing that when we have to find what Nos. we are to *carry* in Addition, &c. we must not now divide by 10, but by the radix of the scale in question.

$$\begin{array}{lrrcrr}
 & \multicolumn{2}{c}{\textit{Addition.}} & & \multicolumn{2}{c}{\textit{Subtraction.}} \\
 & r=4 & r=7 & & r=3 & r=12 \\
\text{Ex. 1.} & 32123 & 65432 & & 201210 & 7t348 \\
 & 21003 & 54321 & & 102221 & 5e6t4 \\
\cline{5-6}
 & 33012 & 43210 & & 21212 & 1t864 \\
 & 22033 & 1444 & & & \\
 & 31102 & 65001 & & & \\
\cline{2-3}
 & 332011 & 226041 & & &
\end{array}$$

Ex. 2. Multiply 68 and 71 in the undenary scale, and prove the truth of the result by comparing it with that obtained by expressing and multiplying these Nos. in the common scale.

$$\text{Here } \left.\begin{array}{r}68\\71\end{array}\right\} \text{ in the undenary} = \left.\begin{array}{r}74\\78\end{array}\right\} \text{ in the denary;}$$

and multiplying in their respective scales, we have

$$\begin{array}{rcrcl}
68 & & 74 & & \\
71 & & 78 & & 11\underline{)\,5772} \\
\cline{1-1}\cline{3-3}
68 & & 592 & & \;\;11\underline{)\,524}\ldots\ldots 8 \\
431 & & 518 & & \;\;\;\;11\underline{)\,47}\ldots\ldots 7 \\
\cline{1-1}\cline{3-3}
4378 & & 5772 & & \;\;\;\;\;\;\;\;4\ldots\ldots 3
\end{array}$$

i. e. 5772 in the denary = 4378 in the undenary, as it should be.

N.B. A No. in any scale may be multiplied by any power of the radix of that scale, by annexing as many cyphers as there are units in its index.

For if $N = a_n r^n + a_{n-1} r^{n-1} + \text{\&c.} + a_0,$

then $N.r^p = a_n r^{n+p} + a_{n-1} r^{n+p-1} + \text{\&c.} + a_0 r^p,$

where the coeffs of r^{p-1}, r^{p-2}, &c. being wanting, we shall have cyphers in the places of the corresponding digits.

This is the case in Arithmetic with any power of 10.

Ex. 3. Divide 14332216 by 6541 in the septenary scale.

Here, since 6 will not go in 1, we shall have to take, as usual, the first *two* figures, 14, of the dividend; only these do not here denote 1.10 + 4, but 1.7 + 4 or 11; and so in other cases.

$$\begin{array}{r@{\,}l}
6541)\,14332216 & (1456 \\
\underline{6541} & \\
44612 & \\
\underline{36124} & \\
54551 & \\
\underline{45665} & \\
55536 & \\
\underline{55536} &
\end{array}$$

Ex. 4. In what scale will the double of the common No. 145 be expressed as 145?

Let r be the radix of the scale; then $r^2 + 4r + 5$ is the No. expressed in this scale by the digits 1, 4, 5; $\therefore$ by the question, $r^2 + 4r + 5 = 290$, whence $r = 15$.

EXAMPLES.

747. Express 1828, 34705 in the septenary scale.

748. Express 300 in the scales of 2, 3, 4, 5, 6.

749. Express 10000 in the scales of 7, 8, 9, 11, 12.

750. Transform 4321 from the quinary to the septenary scale.

751. Transform 27*t* from the undenary scale to the quinary.

752. What are the common Nos. equivalent to the greatest and least Nos. that can be expressed with 4 digits in the scale, whose radix is 6?

753. Take 6 terms of the series 1, 10, 10^2, &c.; express and add them in the senary scale; and reduce the result to the denary.

754. Find the value of $3.9^4 - 4.9^3 + 5.9^2 - 6.9$, by reducing each term to the septenary scale; and prove the result by comparing it with that found in the denary.

755. The No. 4954, expressed in a different scale, becomes 20305; find the radix of the scale.

756. In what scale of Notation does 16000 become 1003000?

757. Multiply the common Nos. 64 and 33 in the binary and quaternary scales, and reduce the products to the denary.

758. Transform 1756 and 345 from the octenary scale to the nonary; multiply them in both scales, and reduce the products to the denary.

759. Divide 51117344 by 675 in the octenary scale, and 29*t*96580 by 2*tt*9 in the duodenary.

760. Extract the square root of 25400544 in the senary scale, and of 32*e*75721 in the duodenary.

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184. *If two Nos.* P *and* Q *have* p *and* q *digits respectively, their product* PQ *will have either* $p+q$ *or* $p+q-1$.

By (178) the greatest and least values

of P are $r^p - 1$, and r^{p-1}, of Q, $r^q - 1$, and r^{q-1},

and $\therefore$ of PQ, $r^{p+q} - (r^p + r^q) + 1$, and r^{p+q-2}.

Hence (since r^p and r^q will each be > 1)

PQ is not $> r^{p+q} - 1$, nor $< r^{p+q-2}$;

but $r^{p+q} - 1$ is the *greatest* No. that has $p + q$ digits, and r^{p+q-2} is the *least* that has $p + q - 1$ digits;

$\therefore$ PQ cannot have more than $p + q$ digits,

nor less than $p + q - 1$.

185. *If* S *be a third number in the same scale having* s *digits, the product* PQS *will have*

$\mathrm{p} + \mathrm{q} + \mathrm{s}$, *or* $\mathrm{p} + \mathrm{q} + \mathrm{s} - 1$, *or* $\mathrm{p} + \mathrm{q} + \mathrm{s} - 2$ *digits.*

For PQ may have $p + q$ or $p + q - 1$ digits (184); but if PQ have $p + q$ digits,
$(PQ)\,S$ will have $(p+q)+s$ or $(p+q)+s-1$ digits;
or if PQ have $p + q - 1$ digits,
$(PQ)\,S$ will have $(p+q-1)+s$ or $(p+q-1)+s-1$ digits;
therefore, on the whole, PQS will have, as stated above,

$$p + q + s, \text{ or } p + q + s - 1, \text{ or } p + q + s - 2 \text{ digits.}$$

186. *On the same supposition, the quotient* $\frac{P}{Q}$ *will have* $\mathrm{p} - \mathrm{q}$ *or* $\mathrm{p} - \mathrm{q} + 1$ *digits.*

For $\frac{P}{Q}$ is greatest when P is the greatest, and Q the least possible, and *vice versâ*:

$$\text{hence } \frac{P}{Q} \text{ not} > \frac{r^p - 1}{r^{q-1}}, \text{ and } \therefore < \frac{r^p}{r^{q-1}}, \text{ or } r^{p-q+1};$$

$$\text{and } \frac{P}{Q} \text{ not} < \frac{r^{p-1}}{r^q - 1}, \text{ and } \therefore > \frac{r^{p-1}}{r^q}, \text{ or } r^{p-q-1},$$

$$\text{and, } \textit{à fortiori}, > r^{p-q-1} - 1.$$

Now r^{p-q+1} is the *least* No. which has $p - q + 2$ digits, and $r^{p-q-1} - 1$ is the *greatest*, which has $p - q - 1$;

$\therefore \frac{P}{Q}$ has less than $p - q + 2$, and more than $p - q - 1$ digits; and will therefore have $p - q$ or $p - q + 1$.

187. Hence, if the No. P have p digits,
P^2 will have $2p$ or $2p - 1$,
P^3 . . . . $3p$ or $3p - 1$ or $3p - 2$, &c.
which gives the No. of digits in any power of a No.

Again, let $\sqrt{P}$ have x digits, then P will have $2x$ or $2x - 1$ digits: but, since P has p digits,

$$\therefore\ 2x = p, \text{ or } 2x - 1 = p,$$

$$\text{and, hence, } x = \frac{p}{2}, \text{ or } \frac{p+1}{2}.$$

So, let $\sqrt[3]{P}$ have x digits, then P will have

$$3x,\ 3x - 1, \text{ or } 3x - 2 \text{ digits};$$

$$\therefore\ 3x = p, \text{ or } 3x - 1 = p, \text{ or } 3x - 2 = p,$$

$$\text{and, hence, } x = \frac{p}{3}, \text{ or } \frac{p+1}{3}, \text{ or } \frac{p+2}{3};$$

and so on for other *roots* of P.

N.B. These values for x, in the case of the square and cube root, correspond exactly to the No. of dots obtained in *pointing;* not, however, that this is assigned as the *reason* for so pointing, which rests on other considerations, not here explained.

188. The following remarks are of importance with respect to decimal fractions in common Arithmetic.

A decimal fraction may be considered as a vulgar fraction, whose den^r is some power of 10, the No. of decimal places pointed off from the right being the same as the index of the den^r. Hence, if P represent the digits, or, as they are called, the *significant part,* of a decimal of p places, its equivalent vulgar fraction $N = \dfrac{P}{10^p}$.

It is obvious that decimals, having the same significant part, P, may differ much in value, in consequence of the difference in the value of p, *i.e.* in the position of their decimal points; thus

$$1.23 = \frac{123}{10^2},\ .0123 = \frac{123}{10^4},\ 12.3 = \frac{123}{10}.$$

189. *To prove the rule for pointing, in Multiplication of Decimals.*

Let M and N be two fractions, which, expressed as decimals, give the significant parts P and Q, with p and q places of decimals respectively; then

$$M = \frac{P}{10^p},\ N = \frac{Q}{10^q},\ \text{and } M \times N = \frac{P}{10^p} \times \frac{Q}{10^q} = \frac{PQ}{10^{p+q}}.$$

Now $\frac{PQ}{10^{p+q}}$ represents a decimal, whose significant part is PQ (the product of the two decimals as whole Nos.) and having $p + q$ decimal places; hence the rule:

Multiply as in whole Nos.; and in the product point off as many decimal places as there are in the Multiplier and Multiplicand together.

190. *To prove the rule for pointing in Division of Decimals.*

Let M, N, P, Q, p, q be the same as before;

$$\text{then } \frac{M}{N} = \frac{P}{10^p} \div \frac{Q}{10^q} = \frac{P}{10^p} \times \frac{10^q}{Q} = \frac{P}{Q} \cdot \frac{10^q}{10^p};$$

$$\text{hence } \frac{M}{N} = \frac{P}{Q} \cdot \frac{1}{10^{p-q}},\ \text{or} = \frac{P}{Q},\ \text{or} = \frac{P}{Q} \cdot 10^{q-p},\ \text{as } p \underset{<}{\overset{>}{=}} q.$$

Now $\frac{P}{Q}$ is the quotient obtained by dividing P by Q, as in whole Nos.; hence the rule:

Divide, as in whole Nos.; then

1. *If the No. of places in the dividend exceed that in the divisor, point off in the quotient a No. of places equal to that excess;*

2. *If the No. in the dividend be the same as that in the divisor, the quotient will have no decimal places;*

3. *If the No. in the dividend fall short of that in the divisor, annex to the quotient a No. of cyphers equal to that defect.*

N.B. Any cyphers, annexed to the dividend in the process of Division, must be reckoned as so many decimal places; thus $1 \div 12.5 = \frac{1.000}{12.5} = .08$.

191. *To prove the rule for reducing a circulating decimal to a vulgar fraction.*

N.B. We here consider only the *fractional* part of a circulating decimal. If there be any figures *before* the decimal point, they should be kept separate, and connected with the vulgar fraction equivalent to the other part, so making a mixed No.

Let N be a circulating decimal, in which P represents the figures *not* recurring, and Q the *period* or recurring part; and let P and Q contain p and q digits respectively.

$$\begin{aligned} \text{Then } N &= .PQQ, \text{ \&c.} \\ \text{and } 10^p . N &= P.QQQ, \text{ \&c.} \\ \text{and } 10^{p+q} . N &= PQ.QQQ, \text{ \&c.} \end{aligned}$$

$$\therefore (10^{p+q} - 10^p)\,N = PQ - P,$$

$$\text{or } N = \frac{PQ - P}{10^{p+q} - 10^p} = \frac{PQ - P}{10^p\,(10^q - 1)}.$$

Hence the rule (observing that $10^q - 1$ will be expressed by q *nines*, and that 10^p is 1 followed by p cyphers)

For the num^r^, set down the decimal to the end of the first period, and subtract from it the non-recurring part; and for the den^r^, set down as many 9's as there are recurring figures, and as many cyphers as non-recurring figures.

192. Let $\frac{a}{b}$ be a proper fraction in *lowest terms.*

Then if b can be put in the form $2^p . 5^q$, *i. e.* the product of any powers of 2 and 5, the fraction may be reduced to a terminating decimal, in which the number of places will be the greater of the two, p and q.

For if $p > q$, then $\frac{a}{2^p . 5^q} = \frac{a . 5^{p-q}}{2^p . 5^p} = \frac{a . 5^{p-q}}{10^p}$, which, expressed as a decimal (188), has p decimal places;

and if $q > p$, then $\frac{a}{2^p . 5^q} = \frac{a . 2^{q-p}}{2^q . 5^q} = \frac{a . 2^{q-p}}{10^q}$, which, expressed as a decimal, has q decimal places.

193. If b be not of the form $2^p . 5^q$, the fraction can only be reduced to a recurring decimal.

For here no factor, by which we could multiply both num^r and den^r, will make the den^r a power of 10; since all powers of 10 contain only factors 2 and 5, whereas the den^r here contains some factor or factors different from these.

194. In such a case, the number of figures in the period must be less than b; for wherever in the division, by which these figures are obtained, the same rem^r is repeated, the figures, of course, are repeated, *i.e.* the *period* begins; now as the rem^{rs}, b being the divisor, can only be 1, 2, 3, &c. $(b - 1)$, it follows that there *might* be $b - 1$ different rem^{rs}, *i.e* there might be $b - 1$ figures in the period, but no more.

195. If b be of the form $2^p . 5^q . c$, where c consists of factors different from 2 or 5, then the fraction may be expressed as a mixed circulator with p or q places (according as $p >$ or $< q$) in the non-recurring part, and not more than $c - 1$ in the period.

MISCELLANEOUS EXAMPLES.

761. The last term of an AR. series is $3\frac{1}{2}$, the common difference $2\frac{1}{4}$, and sum $-22\frac{3}{4}$; find the No. of terms. *Ans.* 7.

762. The sum of £700 was divided among 4 persons, whose shares were in GEOM. PROG., and the difference between the greatest and least was $\frac{37}{12}$ of the difference between the means: find their shares. *Ans.* £108, £144, £192, £256.

763. A hare is 50 leaps before a greyhound, and takes 4 leaps to his 3; but 2 of the greyhound's are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare? *Ans.* 300.

764. The sum of a GEOM. series is $\frac{7}{9}$, the first term 1, and the sum of the second and third $-\frac{2}{9}$; find the No. of terms. *Ans.* 3.

765. The first two terms of an HARM. series are a and b; continue the series both ways to two terms.

Ans. $\frac{ab}{3b-2a}, \frac{ab}{2b-a} \dots \frac{ab}{2a-b}, \frac{ab}{3a-2b}$.

766. Given the coeffs of the 4th and 6th terms of $(1+x)^{n+1}$ equal to one another; find n. *Ans.* 7.

767. If there be 4 Nos., the first 3 in AR., and the last 3 in HARM. PROG,, shew that the product of the first and fourth equals that of the second and third.

768. S, s, are the sums of two infinite GEOM. series, such that $Ss = 1$; the first two terms of the former are 1 and $\frac{1}{3}$, the second term of the latter is $-\frac{1}{2}$; find the second series.

Ans. $1 - \frac{1}{2} + \frac{1}{4} -$ &c.

769. The sum of the first and third terms of a GEOM. series is a, and of the second and fourth, b; find the first term.

Ans. $\frac{a^3}{a^2+b^2}$.

770. *A* and *B* trade with different sums: *A* gains £200; *B* loses £50, and now *A*'s stock : *B*'s :: 2 : ½; but had *A* gained £100 and *B* lost £85, their stocks would have been as 15 : 3¼. Find the original stock of each. *Ans.* *A*'s, £200; *B*'s, £150.

771. There are n AR. means between 1 and 31, and the 7th = $\frac{5}{9}$ of the $(n-1)$th; find n. *Ans.* 14.

772. Given S the sum and s^2 the sum of the squares of an infinite GEOM. series, shew that the common ratio $= \dfrac{S^2-s^2}{S^2+s^2}$.

773. If a, b, c, d are proportionals, and a the greatest of the four, shew that $a + d > b + c$.

774. If the AR. mean between a and b is double the GEOM., prove that $a : b :: 2 + \sqrt{3} : 2 - \sqrt{3}$.

775. The daily wages of 3 workmen were in AR. PROG., and the No. of days they worked was the No. of shillings the second received *per* day; the whole amount of their wages was 7 guineas, and the best workman received 28*s*. more than the worst. What were their daily wages? *Ans.* 5*s*. 7*s*. 9*s*.

776. S_1, S_2, S_3, &c. are the sums of n terms of different AR. series, having the same first term, and common differences 1, 2, 3, &c. respectively: shew that S_1, S_2, &c. are in AR. PROG.; and when that first term is 1, prove that

$$S_1 + S_2 + \&c. + S_n = \frac{n^2(n^2+3)}{4}.$$

777. If x, y, z be respectively the pth, qth, and rth terms of an AR. series, shew that

$$(p-q)\,z + (r-p)\,y + (q-r)\,x = 0.$$

778. The sum of an AR. series is 129, and the sum of twice the same No. of terms is 438; the common difference being 5, find the series. *Ans.* 9, 14, 19, &c.

779. There are two Nos. in the ratio of 2½ : 2, but, when diminished each by 5, they will be in that of 1⅓ : 1; find them.

Ans. 25 and 20.

780. The sum of £210 was divided in GEOM. PROG. among three persons, the first of whom had £90 more than the last; what did they each receive? *Ans.* £120, £60, £30.

798. A farm was rated at 3*s*. an acre, and the tenant, on receiving back at his rent-day 10 per cent. of his rent, found that the sum returned amounted to £6 more than the whole rate. The next year the rates were doubled, and he received back 15 per cent. of his rent; but he now found that the sum returned only just paid for the whole rate. What was the rent of the farm?

Ans. £240.

799. A pedestrian, finding that he could walk forwards four times as fast as he could backwards, undertook to walk a certain distance ($\frac{1}{4}$ of it backwards) in a certain time. But, the ground being bad, he found that his rate *per* hour backwards was $\frac{1}{5}$ of a mile less than he had supposed, and that, to have won his wager, he must have walked forwards two miles an hour faster than he did. What was his rate backwards? *Ans.* One mile.

800. *A* and *B* start to run a race to a certain post and back again. *A* returning meets *B* 90 yards from the post, and reaches the starting-place 3′ before him. Had he returned at once, he would have met *B* at $\frac{1}{6}$ of the distance between the post and the starting-place. Find the length of the course.

Ans. 1080 yards.

ANSWERS TO THE EXAMPLES.

1. 40.
2. 12.
3. – 7.
4. 14.
5. 34.
6. – 178.
7. 255.
8. 920.
9. 192.
10. – 930.

11. 11.
12. 1.
13. 0.
14. 94.
15. 94.
16. – 64.
17. 16.
18. 264.
19. 5.
20. 7.

21. 37.
22. 12.
23. 6.
24. 21.
25. 22.
26. 7.
27. 13.
28. 71.
29. 18.
30. 26.

31. 10.
32. 24.
33. 35.
34. 9.
35. 7200.
36. 135.
37. 8.
38. 120.
39. 384.
40. 4.

41. $15a + 3b - 6c + 6d$.
42. $14x - 9y + 10z - 12$.
43. $23a^2 - 26ab + 14b^2$.
44. $6by - 7cz$.
45. $5x^3 + 50x^2y - 14xy^2 + 4y^3$.
46. $2x^2 + 2y^2 + 2z^2$.
47. $-9x^3 + 2ax^2 - 31a^2x + 16a^3$.
48. $a^3 + b^3 + c^3 + 6abc$.
49. $6x^3 + 4y^3 + z^3 - 24xyz$.
50. $x^4 + y^4 + z^4$.

51. $a - 3b + 3c.$ 52. $-2x^2 - 7xy + 3y^2.$
53. $4ax - 9by + 2cz.$ 54. $5x^2 - 5x + 5.$
55. $7a^2 - 3a + 4b^2 - 7ab + 2c^2 - 6bc.$
56. $-x^3 - 6x^2y + 6 - 4y^3.$
57. $3x^2 + 13xy - y^2 - 16xz - 13yz.$
58. $x^2 + xy + y^2.$ 59. 0.
60. $3a^4 - 4a^3b - 4ab^3 + 2b^4.$

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61. $4a^3 - 4a^2c.$ 62. $x^2 - 3y^2 - 3z^2.$
63. $2ax^2 + 2by^2 + 2cz^2.$ 64. $a^2 - 3b^2 + 3c^2.$
65. $2ab + 4b^2.$ 66. 0.
67. $-4c + 4d.$

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68. abx^3y^4; $-mnx^5$; $2a^2cx^2y$; ab^2c^2; a^2bc^2; $-x^3y^3.$
69. $x^3 - x^2y + xy^2$; $-a^3x + a^2x^2 - ax^3$; $-abx^3 + a^2bx^2 - ab^2x$;
$x^4y - 3x^3y^2 + 3x^2y^3 - xy^4.$
70. $2a^2 + 7ab + 3b^2$; $ac - bc - ad + bd.$
71. $6x^2 + 13xy + 6y^2$; $6a^2b^2 - ab^3 - 12b^4.$
72. $81x^4 - y^4.$
73. $a^5 + 32b^5$, and $a^5 - 4a^4b + 8a^3b^2 - 16a^2b^3 + 32ab^4 - 32b^5.$
74. $x^4 - 4a^3x + 3a^4.$
75. $27a^3 + b^3 + 8 - 18ab.$
76. $x^3 - y^3 + z^3 + 3xyz.$ 77. $a^6 - 1.$
78. $a^3 - 8b^3 - 27c^3 - 18abc.$
79. $x^8 + x^6y^2 - x^2y^6 - y^8.$ 80. $a^6 + 2a^3b^3 + b^6.$

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81. $x^4 - 13x^2 + 36.$ 82. $x^4 - 5a^2x^2 + 4a^4.$
83. $4a^4 - 17a^2b^2 + 4b^4.$
84. $x^5 + 2x^4 - 17x^3 - 34x^2 + 16x + 32.$

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85. bc^2; $5xy^2$; $-35bx.$
86. $3xy - 2xz + 3yz$; $-a^2b^2 + 7abc^2 - 4c^4$;

$$\frac{a^2x^2}{b} - 3\,ax + 3by - \frac{b^2y^2}{a}.$$

87. $-\frac{2m}{3n}+1-\frac{4m^2}{3n^2}+\frac{n}{3m}$; $\frac{3a^2}{2b^2}-\frac{5a}{2b}+3+\frac{b}{2a}-\frac{2b^2}{a^2}$.
88. $3a-2b$; $3x+2y$.
89. $x^3+3x^2y+9xy^2+27y^3$. 90. $2ab-3b^2$.
91. $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$.
92. $x^{15}-x^{12}y^2+x^9y^4-x^6y^6+x^3y^8-y^{10}$.
93. $x^4-2x^3y+4x^2y^2-8xy^3+16y^4$.
94. $1-2x+3x^2-4x^3+5x^4$.
95. $x^2+2xy+3y^2$.
96. $a^3+2a^2b+3ab^2+4b^3$.
97. $x^4+2x^3+3x^2+2x+1$.
98. $x^3+2ax^2+2a^2x+a^3$.
99. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.
100. $1-x+2y+x^2+2xy+4y^2$.
101. $a^2+b^2+c^2-ab-ac-bc$.
102. $x^2+4y^2+9z^2+2xy+3xz-6yz$.
103. $1+5x+15x^2+45x^3+135x^4+405x^5+\frac{1215x^6}{1-3x}$.
104. $1+2x+3x^2+4x^3+5x^4+6x^5+\frac{7x^6-6x^7}{1-2x+x^2}$.

105. $3x^2$; $2ab^2$; $4y^3z^3$. 106. ax; a; x.

107. $3x-2$. 108. $2x+3$.
109. $3x+5$. 110. $8x^2+14x-15$.
111. $4x-5$. 112. x^2+2x-3.

113. $a+x$. 114. $x-1$.
115. $2x^2+4x+2$. 116. $y-2$.
117. $x-2a$. 118. $x+3$.
119. $3x+9$. 120. a^2+ab.
121. a^4-ab^2. 122. $x^2-2xy+y^2$.

123. $2x+3$. 124. $3x-2$.
125. $x-1$. 126. x^2+4x+4.

127. $x - 3$.

128. $x - y$.

129. $x + 3$.

130. $5x^2 - 1$.

131. $3x^2 - 2xy + y^2$.

132. $2x^3 + 2x^2y - xy^2$

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133. $12a^2b^2c$; $36x^3y^3$; $ax^2y - axy^2$; $ab^2 - ad^2$.

134. $120a^4b^2$; $10a^5b^5$; $1800a^3x^3$.

135. $6(a^2 - b^2)$; $12a(a^2 - 1)$; $120xy(x^2 - y^2)$.

136. $24a^2b^2(a^2 - b^2)$; $36xy^2(x^2 - y^2)$.

137. $\frac{7x}{5a}$; $\frac{a+y}{a}$; $\frac{a^2 - 3ab}{4b^2 + 2ab}$.

138. $\frac{x^2 - 3y^2}{xy - 2y^2}$; $\frac{2mn}{m+n}$; $\frac{3abc}{a+b+c}$.

139. $\frac{x-1}{x+1}$.

140. $\frac{x-1}{x+2}$.

141. $\frac{3a - 2x}{5a + 3x}$.

142. $\frac{x+4}{x^2 - 2x + 1}$.

143. $\frac{x^2 + 4x + 4}{x^2 + x + 1}$.

144. $\frac{x^2 + x - 2}{x^2 + 5x + 5}$.

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157. $\frac{bcx}{abc}$, $\frac{acy}{abc}$, and $\frac{abz}{abc}$; $\frac{6cx^2}{12abc}$, $\frac{4by^2}{12abc}$, and $\frac{3az^2}{12abc}$; $\frac{40b^3x^2y}{60a^3b^3}$, $\frac{45ab^2x^3}{60a^3b^3}$, $\frac{48a^2by^3}{60a^3b^3}$, and $\frac{50a^3xy^2}{60a^3b^3}$.

158. $\frac{16ax^2 - 16bx^2}{12(a^2 - b^2)}$, $\frac{15ay^2 + 15by^2}{12(a^2 - b^2)}$, and $\frac{2xy}{12(a^2 - b^2)}$.

159. $\frac{a^2x^2 - b^2x^2}{a^4 - b^4}$ and $\frac{a^2y^2 + b^2y^2}{a^4 - b^4}$; $\frac{a^2 + 2ax + x^2}{a^2 - x^2}$ and $\frac{a^2 - 2ax + x^2}{a^2 - x^2}$.

160. $\frac{a - x}{4a^3(a^2 - x^2)}$, $\frac{a + x}{4a^3(a^2 - x^2)}$, and $\frac{2a}{4a^3(a^2 - x^2)}$.

161. a; b; $\frac{ab}{a-b}$.

162. $\frac{a^2+b^2}{a^2-b^2}$; $\frac{a^2+b^2}{a^2-b^2}$; $\frac{a^2-ab+b^2}{a^2-b^2}$.

163. $\frac{2a^2-2ab+2b^2}{a^2-b^2}$; $\frac{x-y}{x}$. 164. $\frac{1}{x^2(x^2-1)}$; 0.

165. $\frac{2a^3}{a^4-x^4}$. 166. $\frac{2x}{x+y}$.

167. $\frac{2x^4+x^2y^2-4xy^3-y^4}{x^4-y^4}$. 168. $\frac{y}{x+y}$.

169. $\frac{x-3x^2+3x^3}{(1-x)^3}$. 170. $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

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171. $\frac{ax+x^2}{ab-bx}$ and $\frac{a^3+2a^2x+2ax^2+x^3}{a^3-2a^2x+2ax^2-x^3}$.

172. $\frac{a^4-x^4}{a^2x}$ and $\frac{3a^2(a-b)}{b}$.

173. $\frac{2ax^2(x-y)}{c}$. 174. $\frac{a^2-b^2}{a^4-4b^4}$ and $\frac{x}{x^2+y^2}$.

175. $\frac{a^2+b^2}{a^2-2ab+b^2}$ and 1.

176. $\frac{3(a-b)}{2b(a+b)}$. 177. $\frac{x^2+b^2}{x}$.

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178. $\frac{4-3x}{10}$; $\frac{3x}{15-2x}$; $\frac{6-2x}{2x+5}$; $\frac{18x+14}{21}$.

179. $\frac{10-13x}{6}$; $x-1$; $\frac{20-3x}{2x-25}$; $\frac{14-20x}{9(x+1)}$.

180. $\frac{1}{x}$; 1; $\frac{a^2+x^2}{2ax}$.

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181. $4a^2b^4$, $-27a^6b^6c^{12}$, $\dfrac{81a^4b^8}{256c^{12}}$, $\dfrac{x^{10}y^{15}z^{20}}{32}$, $\dfrac{a^{12}b^{24}c^{18}}{729d^{12}}$.

182. $4a^2 - 12ab + 9b^2$, and $9x^2 + 24xy + 16y^2$.

183. $9 - 12x + 10x^2 - 4x^3 + x^4$, and $1 - 2x + 3x^2 - 2x^3 + x^4$.

184. $1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6$.

185. $4x^2 - 12xy + 9y^2 + 16xz - 24yz + 16z^2$.

186. $9a^2x^2 + 12abxy + 4b^2y^2 + 6acxz + 4bcyz + c^2z^2$.

187. $\dfrac{a^2}{b^2} + 2 + \dfrac{b^2}{a^2}$, and $\dfrac{4x^2}{9y^2} - 2 + \dfrac{9y^2}{4x^2}$.

188. $\dfrac{9x^4}{y^4} - \dfrac{12x^3}{y^3} + \dfrac{10x^2}{y^2} - \dfrac{16x}{y} + 27 - \dfrac{16y}{x} + \dfrac{10y^2}{x^2} - \dfrac{12y^3}{x^3} + \dfrac{9y^4}{x^4}$.

189. $x^6 - 3x^2 + \dfrac{3}{x^2} - \dfrac{1}{x^6}$, and $x^6 + 3x^4 + 6x^2 + 7 + \dfrac{6}{x^2} + \dfrac{3}{x^4} + \dfrac{1}{x^6}$.

190. $a^6 - 6a^5b + 21a^4b^2 - 44a^3b^3 + 63a^2b^4 - 54ab^5 + 27b^6$.

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191. $x^4 - 8x^3 + 24x^2 - 32x + 16$, $x^4 + 8x^3 + 24x^2 + 32x + 16$, and $x^4 - 4x^2 + 6 - \dfrac{4}{x^2} + \dfrac{1}{x^4}$.

192. $x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$.

193. $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$.

194. $\dfrac{a^6}{b^6} - \dfrac{6a^4}{b^4} + \dfrac{15a^2}{b^2} - 20 + \dfrac{15b^2}{a^2} - \dfrac{6b^4}{a^4} + \dfrac{b^6}{a^6}$.

195. $x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \dfrac{56}{x^2} + \dfrac{28}{x^4} + \dfrac{8}{x^6} + \dfrac{1}{x^8}$.

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196. $\pm 2ab^2c^3$, $\pm 7x^2y^3z$, and $\pm 10a^4b^6c^8$.

197. $\pm \dfrac{3ax^2y^3}{5z}$, $\pm \dfrac{7xy^2}{8a}$, and $\pm \dfrac{5x^3y^5}{4ab^2}$.

198. $\pm \dfrac{a^2x^2y}{2}$, $-\dfrac{2ay^2}{3x^3}$, and $\dfrac{4b^2c^3}{5a^4}$.

199. $-\frac{6abc^5}{7}$, $\pm\frac{2xy^2}{5a^3}$, and $\pm\frac{3ab^2c^3}{4x^4}$.

200. $\frac{2ab^2}{c^3}$, and $\pm\frac{2x^2y}{3z^3}$.

201. $2x + y$ and $5a - 3b$.
202. $7ab - a^2$ and $5x^2 + 3xy$.
203. $4xy + 5yz$ and $5a^2bc + c^4$.
204. $3x^2 + \frac{1}{6x^2}$ and $5xy - \frac{2x}{y}$.
205. $\frac{7x^2y}{2} - \frac{xy^2}{14}$ and $\frac{1}{2} + ab$.

206. $1 + 2x + 3x^2$. 207. $3x^2 + 2x + 3$.
208. $3a + 2b + c$. 209. $x^2 - 4xy + 4y^2$.
210. $\frac{x}{2} - \frac{y}{3} + \frac{z}{4}$. 211. $\frac{a}{x} - \frac{2b}{y} + \frac{3c}{z}$.
212. $\frac{a}{b} + 1 + \frac{b}{a}$. 213. $\frac{x^2}{4} + \frac{3x}{y} + \frac{6}{y^2}$.
214. $x^3 - 2x^2y + 2xy^2 - y^3$.
215. $1 - 3x + 3x^2 - x^3$. 216. $2 - 3a - a^2 + 2a^3$.
217. $1 - x$. 218. $2a - 3b$.
219. $\frac{a}{b} - \frac{2b}{a}$. 220. $a - b$.

221. 421, 3789, 75.78, 15.156.
222. 347, 694, 2082, 4164, 832.8.
223. 1046, 20.92, 2568.2, 11367.
224. 1011, 20.22, 4.044, 8088.
225. 32, 64, 128, 256. 226. 48, 9.6, 192, 3.84.
227. 117, 2.34, 46.8, 936.

228. 32.1, 6.42, .512, 1.284.
229. 1.581138, .447213, .031622, .070710.
230. 1.357, .584, .215, .368.

231. $x = 5$.
232. $x = 2$.
233. $x = 3$.
234. $x = \frac{4}{7}$.
235. $x = -\frac{1}{2}$.
236. $x = 2$.
237. $x = 1$.
238. $x = 4$.
239. $x = -4$.
240. $x = \frac{5}{6}$.

241. $x = 5$.
242. $x = 7$.
243. $x = 4$.
244. $x = 1$.
245. $x = 5$.
246. $x = 42$.
247. $x = 12$.
248. $x = 3\frac{4}{7}$.
249. $x = 7$.
250. $x = 12$.
251. $x = -8$.
252. $x = 7$.
253. $x = 4$.
254. $x = 8$.
255. $x = -\frac{3}{2}$.
256. $x = 5\frac{1}{10}$.

257. $x = 4$.
258. $x = \frac{2}{5}$.
259. $x = -1$.
260. $x = 7$.
261. $x = 8$.
262. $x = 3\frac{3}{4}$.
263. $x = 2$.
264. $x = 11$.

265. $x = 1\frac{1}{3}$.
266. $x = -\frac{3}{5}$.
267. $x = -1\frac{4}{5}$.
268. $x = 1$.
269. $x = 4$.
270. $x = 20$.
271. $x = 8$.
272. $x = 14$.
273. $x = -107$.
274. $x = 4$.
275. $x = \frac{3}{2}$.
276. $x = 4$.

277. 7*s*.
278. 9.
279. 35 and 13.
280. 15.

281. 60, 90. 282. 16.
283. £5. 284. 15 yds. and 5 yds.
285. 88 years. 286. £36, £12, £16.
287. £5. 8*s*. 288. £45, £57, £63, £65.
289. 22, 7, and 12 gallons.
290. 4*s*. 8*d*. 291. 3000.
292. £125. 293. $23\frac{1}{3}$ days.
294. £300, £3600, £2400, £1800.
295. *A*, £72; *B*, £108. 296. 180000.
297. $x = 1$, $y = 1$. 298. $x = 1$, $y = -1$.
299. $x = 5$, $y = 2$. 300. $x = 2$, $y = 1$.
301. $x = 6$, $y = 7$. 302. $x = 10$, $y = 24$.
303. $x = 144$, $y = 216$. 304. $x = 2$, $y = 3$.
305. $x = 3$, $y = 5$. 306. $x = 3$, $y = 2$.
307. $x = 6$, $y = 8$. 308. $x = 5$, $y = 2$.
309. $x = 5$, $y = 5$. 310. $x = 7$, $y = 9$.
311. $x = 21$, $y = 20$. 312. $x = -2$, $y = -\frac{1}{2}$.

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313. $x = 1$, $y = 2$, $z = 3$. 314. $x = 7$, $y = 10$, $z = 9$.
315. $x = -5$, $y = 6$, $z = -2$.
316. $x = 5$, $y = 6$, $z = 7$. 317. $x = 4$, $y = -5$, $z = 6$.
318. $x = 2$, $y = -3$, $z = 4$. 319. $x = 12$, $y = 12$, $z = 12$.
320. $x = 5$, $y = 7$, $z = -3$.

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321. $\frac{3}{15}$. 322. 40 half-guineas, 21 crowns.
323. *A*, 5*s*.; *B*. 3*s*.
324. Of a cow, £12; of a horse, £24.
325. 17 and 13 yards. 326. 48.
327. 108 square feet. 328. 18*s*.; 10 passengers.
329. 2*s*. 6*d*. 330. 4 hours and 6 hours.

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331. $x = \pm 2$.
332. $x = \pm 3$.
333. $x = \pm 1$.
334. $x = \pm \frac{1}{2}$.
335. $x = \pm 2\frac{1}{3}$.
336. $x = \pm \frac{1}{2}$.
337. $x = \pm \frac{2}{3}$.
338. $x = \pm 5$.
339. $x = \pm 2$.
340. $x = \pm 2$.
341. $x = \pm 3$.
342. $x = \pm 5$.

343. $x = 4$ and -2.
344. $x = -1$ and -9.
345. $x = 20$ and -6.
346. $x = 7$ and 5.
347. $x = 8$ and -40.
348. $x = 10$ and -110.

349. $x = 1$ and -8.
350. $x = 17$ and -4.
351. $x = -5$ and -20.
352. $x = -1$ and -12.
353. $x = 1$ and -20.
354. $x = 25$ and -136.

355. $x = 6$ and $-5\frac{2}{3}$.
356. $x = 6$ and $-4\frac{1}{2}$.
357. $x = 8\frac{3}{5}$ and -10.
358. $x = 14$ and $-10\frac{2}{7}$.
359. $x = 12$ and $-12\frac{1}{12}$.
360. $x = 13$ and $-11\frac{4}{13}$.

361. $x = 10$ and 2.
362. $x = 3$ and -1.
363. $x = 2$ and $-\frac{3}{7}$.
364. $x = 1\frac{2}{3}$ and $-1\frac{1}{2}$.
365. $x = 1\frac{1}{2}$ and $-\frac{15}{22}$.
366. $x = 7$ and $-1\frac{1}{2}$.
367. $x = 2$ and $\frac{12}{25}$.
368. $x = 2$ and -3.
369. $x = 2$ and $\frac{1}{2}$.
370. $x = 3$ and $-\frac{4}{5}$.
371. $x = 6$ and $3\frac{1}{13}$.
372. $x = 11$ and -13.
373. $x = 5\frac{2}{5}$ and 5.
374. $x = 21$ and 5.
375. $x = 6$ and $-4\frac{3}{4}$.
376. $x = 7$ and $-1\frac{3}{7}$.
377. $x = 10\frac{2}{3}$ and 1.
378. $x = 3$ and $-8\frac{7}{10}$.

379. $x = \pm 2$ and ± 1.
380. $x = \pm 2$ and $\pm \frac{1}{2}$.
381. $x = \pm 5$ and ± 2.
382. $x = \pm 1$ and ± 3.
383. $x = \pm 2$ and ± 1.
384. $x = \pm 1$ and $\pm \frac{1}{2}$.

385. $x^2 - 4x - 21 = 0.$ 386. $6x^2 + 5x - 6 = 0.$

387. $16x^4 - 153x^2 + 81 = 0.$

388. $x^4 - 6x^3 + 11x^2 - 6x = 0.$

389. $4x^5 + 3x^4 - 17x^3 - 12x^2 + 4x = 0.$

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390. $x = 7,\ y = \pm 4.$ 391. $\left.\begin{array}{r} x = 4,\ y = -3 \\ x = -3,\ y = 4 \end{array}\right\}.$

392. $\left.\begin{array}{r} x = 4,\ y = 3 \\ x = -1\tfrac{19}{25},\ y = 4\tfrac{17}{25} \end{array}\right\}.$ 393. $\left.\begin{array}{r} x = 8,\ y = 2\tfrac{1}{2} \\ x = -2\tfrac{1}{2},\ y = -8 \end{array}\right\}.$

394. $\left.\begin{array}{r} x = 6,\ y = 5 \\ x = -6,\ y = -5 \end{array}\right\}.$ 395. $\left.\begin{array}{r} x = 5,\ y = 3 \\ x = \tfrac{3}{4},\ y = -1\tfrac{1}{4} \end{array}\right\}.$

396. $\left.\begin{array}{r} x = 5,\ y = 3 \\ x = 1\tfrac{7}{10},\ y = -\tfrac{3}{10} \end{array}\right\}.$ 397. $\left.\begin{array}{l} x = 3,\ y = 4 \\ x = -\tfrac{12}{11},\ y = -\tfrac{26}{11} \end{array}\right\}.$

398. $\left.\begin{array}{l} x = 4,\ y = 2 \\ x = 2,\ y = 4 \end{array}\right\}.$ 399. $\left.\begin{array}{l} x = 10,\ y = 15 \\ x = -10\tfrac{7}{9},\ y = -16\tfrac{1}{6} \end{array}\right\}.$

400. $\left.\begin{array}{l} x = 3,\ y = 2 \\ x = -2,\ y = -3 \end{array}\right\}.$ 401. $\left.\begin{array}{l} x = 5,\ y = 4 \\ x = 4,\ y = 5 \end{array}\right\}.$

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402. ± 12, ± 15. 403. ± 10, ± 16.

404. ± 4, ± 12. 405. 15 yds.; 25 yds.

406. 8 and 6, or 56 and − 42.

407. 27 yards. 408. 4550.

409. 24, or − 3. 410. 4, or $-1\tfrac{1}{3}$.

411. 40 yards by 24. 412. 9, 12, 15.

413. 12 and 7, or $-9\tfrac{1}{2}$ and $-14\tfrac{1}{2}$.

414. 10 yds.; 16 yds. 415. 3 inches.

416. 26 feet; 38 feet. 417. 16.

418. 49 yards; £3 per yard.

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419. $\left.\begin{array}{r} x = 23 \\ y = 2 \end{array}\right\}\ \left.\begin{array}{r} x = 31 \\ y = 2 \end{array}\right\}\ \left.\begin{array}{r} 2 \\ 5 \end{array}\right\}\ \left.\begin{array}{r} x = 30 \\ y = 1 \end{array}\right\}\ \left.\begin{array}{r} 15 \\ 8 \end{array}\right\}\ \left.\begin{array}{r} 0 \\ 15 \end{array}\right\}.$

420. $\left.\begin{array}{l} x = 5 \\ y = 12 \end{array}\right\}$ $\left.\begin{array}{l} x = 49 \\ y = 22 \end{array}\right\}$ $\left.\begin{array}{l} x = 11 \\ y = 3 \end{array}\right\}.$

421. 4; 5; 5; 9. 422. 5.

423. 5; 3. 424. 4.

425. *A* gives 14 pieces, *B* 9.

426. 8; 16. 427. 21, 12.

428. 4. 429. 59.

430. 3, 21, 16; or 6, 2, 32.

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443. $x^{\frac{3}{2}} + 2x^{\frac{4}{3}} + 3x^{\frac{5}{4}} + 4x^{\frac{2}{3}}$.

444. $a^{\frac{4}{3}} + 4ab^{\frac{2}{3}} + 6a^{\frac{2}{3}}b^{\frac{4}{3}} + 4a^{\frac{1}{3}}b^{2} + ab^{\frac{8}{3}}$.

445. $a^{\frac{5}{2}} + 2a^{2}b^{\frac{1}{3}} + 4ab^{\frac{2}{3}} + 8a^{\frac{1}{2}}b^{2} + 5ab^{2}$.

446. $a^{-1} + 2b^{-2} + 3c^{-3}$; $ab^{-1} + a^{-1}b$;

$a^{3}b^{-3} + 3a^{2}b^{-1} + 4a^{-2}b + 2a^{-3}b^{2}$;

and $\frac{1}{a} + \frac{2}{b^{2}} + \frac{3}{c^{3}}$; $\frac{1}{a^{-1}b} + \frac{1}{ab^{-1}}$;

$$\frac{1}{a^{-3}b^{3}} + \frac{3}{a^{-2}b} + \frac{4}{a^{2}b^{-1}} + \frac{2}{a^{3}b^{-2}}.$$

447. $\frac{a^{3}b^{-2}c^{-2}}{3} + 4a^{-2}b^{-1}c^{2} + 2a^{-1}bc + \frac{a^{-1}b^{-1}c^{-1}}{3}$;

$$\frac{1}{3a^{-3}b^{2}c^{2}} + \frac{4}{a^{2}bc^{-2}} + \frac{2}{ab^{-1}c^{-1}} + \frac{1}{3abc}.$$

448. $\frac{abc^{-\frac{1}{3}}}{2} + \frac{2a^{-\frac{3}{2}}b^{2}c^{2}}{3} + \frac{3a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-1}}{4} + 5a^{-1}b^{-\frac{3}{4}}c$;

$$\frac{1}{2a^{-1}b^{-1}c^{\frac{1}{3}}} + \frac{2}{3a^{-\frac{3}{2}}b^{-2}c^{-2}} + \frac{3}{4a^{\frac{2}{3}}b^{\frac{1}{3}}c} + \frac{5}{ab^{\frac{3}{4}}c^{-1}}.$$

449. $\sqrt{a} + 2\sqrt[3]{a^{2}} + 3\sqrt[4]{a^{3}} + 4\sqrt[5]{a}$.

450. $\sqrt[4]{a^3} + \sqrt{ab} + \sqrt[4]{ab^3} + \sqrt{b^3}$.

451. $\dfrac{\sqrt[3]{a^2b}}{2\sqrt{c}} + \dfrac{2\sqrt[4]{ac^3}}{3\sqrt{b^3}} + \dfrac{\sqrt[5]{ab^2c^3}}{4} + \dfrac{\sqrt[6]{bc^5}}{5\sqrt[4]{a^3}}$.

452. $\dfrac{bc}{a} + \dfrac{2ac}{b^2} + \dfrac{3}{abc} + \dfrac{4c^3}{ab^2}$.

453. $\dfrac{c}{a^2b^2} + 2abc + \dfrac{3a^3}{bc^2} + ab^2c^3$.

454. $\dfrac{1}{\sqrt[5]{a^2}} + \dfrac{\sqrt[3]{b}}{a^2} + \dfrac{\sqrt{a^3}}{\sqrt[3]{b^2}} + ab + \dfrac{\sqrt{a}}{\sqrt[3]{b^4}} + \dfrac{1}{\sqrt{b^5}}$.

455. $x^2 + 6xz^{\frac{1}{3}} + 9z^{\frac{2}{3}} - 4y$.

456. $a - b^2$.

457. $a^3 - 64b^2$.

458. $a^3 - a^{\frac{4}{3}} + 2a^{\frac{1}{3}} - 2 - a^{-\frac{2}{3}} + a^{-3}$.

459. $x + x^{\frac{3}{4}}y^{-\frac{1}{4}} - x^{\frac{1}{4}}y^{-\frac{3}{4}} - y^{-1}$.

460. $8x^{\frac{3}{4}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 2x^{\frac{1}{4}}y + y^{\frac{3}{2}}$.

461. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

462. $a^{-\frac{5}{2}} - 2a^{-2}b^{\frac{1}{3}} + 4a^{-\frac{3}{2}}b^{\frac{2}{3}} - 8a^{-1}b + 16a^{-\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$.

463. $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$.

464. $4a - 2a^{\frac{1}{2}}b^{-\frac{1}{2}} + 2a^{\frac{1}{2}}c^{\frac{1}{3}} + b^{-1} + b^{-\frac{1}{2}}c^{\frac{1}{3}} + c^{\frac{2}{3}}$.

465. $ab^{-3} + 3a^{\frac{1}{3}}b^{-1} + 3a^{-\frac{1}{3}}b + a^{-1}b^3$;

$\dfrac{8x^2y^{-1}}{27} - 2x + \dfrac{9y}{2} - \dfrac{27x^{-1}y^2}{8}$.

466. $a^2 - 6a^{\frac{5}{3}}b^{\frac{1}{6}} + 21a^{\frac{4}{3}}b^{\frac{1}{3}} - 44ab^{\frac{1}{2}} + 63a^{\frac{2}{3}}b^{\frac{2}{3}} - 54a^{\frac{1}{3}}b^{\frac{5}{6}} + 27b$.

467. $x^3 - 4x^{\frac{9}{4}}y^{\frac{5}{2}} + 6x^{\frac{3}{2}}y^5 - 4x^{\frac{3}{4}}y^{\frac{15}{2}} + y^{10}$.

468. $a^2b^{-2} - 4ab^{-1} + 6 - 4a^{-1}b + a^{-2}b^2$.

469. $ab^{-1} + 1 + a^{-1}b$.

470. $a^{\frac{2}{3}} - \dfrac{3a^{\frac{1}{3}}}{2} + 3 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}}$.

471. $xy^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{2}{3}}$. 472. $2x^{\frac{3}{2}} - 3y^{\frac{3}{4}}$.

473. $64^{\frac{1}{4}}$, $81^{\frac{1}{5}}$, $(\frac{1}{9})^{\frac{1}{3}}$, $(\frac{1}{8})^{\frac{1}{4}}$, $(\frac{3}{2})^{\frac{1}{4}}$, $8^{\frac{1}{5}}$.

474. $25^{\frac{1}{2}}$, $\left(\dfrac{25}{4}\right)^{\frac{1}{2}}$, $\left(\dfrac{4a^2}{9}\right)^{\frac{1}{2}}$, $\left(\dfrac{a^2 + 2ab + b^2}{4}\right)^{\frac{1}{2}}$;

$125^{\frac{1}{3}}$, $\left(\dfrac{125}{8}\right)^{\frac{1}{3}}$, $\left(\dfrac{8a^3}{27}\right)^{\frac{1}{3}}$, $\left(\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{8}\right)^{\frac{1}{3}}$

475. $\left(\dfrac{1}{729}\right)^{\frac{1}{3}}$, $\left(\dfrac{27}{1000}\right)^{\frac{1}{3}}$, $\left(\dfrac{1}{a^6}\right)^{\frac{1}{3}}$, $\left(\dfrac{a^3}{b^3c^6}\right)^{\frac{1}{3}}$;

$\left(\dfrac{1}{6561}\right)^{\frac{1}{4}}$, $\left(\dfrac{81}{10000}\right)^{\frac{1}{4}}$, $\left(\dfrac{1}{a^8}\right)^{\frac{1}{4}}$, $\left(\dfrac{a^4}{b^4c^8}\right)^{\frac{1}{4}}$.

476. $\sqrt{125}$, $\sqrt{3}$, $\sqrt{12}$, $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{320}$.

477. $\sqrt[3]{54}$, $\sqrt[3]{256}$, $\sqrt[4]{2048}$, $\sqrt[4]{3}$, $\sqrt[3]{\frac{2}{3}}$, $\sqrt[5]{\frac{1}{18}}$.

478. $\sqrt{4a}$, $\sqrt{98a^3x}$, $\sqrt{2ab}$, $\sqrt{6a^3x}$, $\sqrt[3]{\dfrac{4a^2}{9b^2}}$.

$\sqrt[3]{\dfrac{2a}{3}}$, $\sqrt{a^2 - x^2}$, $\sqrt{\dfrac{a+b}{a-b}}$.

479. $3\sqrt{5}$, $5\sqrt{5}$, $36\sqrt{3}$, $3\sqrt[3]{5}$, $18\sqrt[3]{2}$, 6.

480. $4\sqrt[4]{2}$, $8\sqrt[3]{2}$, $6\sqrt[5]{48}$, $\frac{2}{3}\sqrt{2}$, $\frac{2}{27}\sqrt{2}$, $\frac{2}{9}\sqrt[5]{2}$.

481. $\frac{1}{2}\sqrt{6}$, $\sqrt[3]{12}$, $\sqrt[4]{54}$, $\frac{3}{7}\sqrt{21}$, $\frac{3}{2}\sqrt[3]{150}$, $\sqrt[4]{375}$.

482. $2\sqrt{3}$, $15\sqrt{3}$, $\frac{7}{2}\sqrt{3}$, $\frac{4}{45}\sqrt{3}$, $\frac{1}{2}\sqrt{3}$, $\frac{1}{6}\sqrt{3}$.

483. $\sqrt{108}$ and $\sqrt{112}$; $\sqrt[3]{81}$ and $\sqrt[3]{80}$;
$\sqrt[3]{120}$, $\sqrt[3]{128}$, and $\sqrt[3]{135}$.

484. $\sqrt[6]{125}$ and $\sqrt[6]{121}$; $\sqrt[4]{\frac{1}{4}}$ and $\sqrt[4]{\frac{1}{3}}$;
$\sqrt[6]{125}$, $\sqrt[6]{144}$, and $\sqrt[6]{162}$.

485. $\sqrt{2}$. 486. $\frac{29}{2}\sqrt{3}$.

487. $9\sqrt[3]{9}$. 488. $3\sqrt[3]{5}$.

489. $24\sqrt{3}$; $120\sqrt{3}$; 36.

490. $216\sqrt[12]{6}$; $288\sqrt[12]{72}$. 492. 16.

491. $5-\sqrt{6}$; $6\sqrt{3}+3\sqrt{30}$.

493. $\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{3}$; $\frac{1}{3}\sqrt{6}+\frac{1}{2}\sqrt[6]{32}+\frac{1}{6}\sqrt[4]{120}$.

494. $\frac{2\sqrt{2}+\sqrt{3}}{5}$; $\sqrt{5}+1$; $\sqrt{5}-\sqrt{2}$.

495. $\frac{58+8\sqrt{7}}{27}$; $\frac{8\sqrt{5}+23}{19}$; $\frac{3-\sqrt{6}}{6}$.

496. $4+\sqrt{2}$; $\frac{7+3\sqrt{5}}{2}$; $\frac{7\sqrt{14}-13}{11}$.

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497. $\sqrt{3}+1$. 498. $3+\sqrt{2}$.

499. $\sqrt{5}-1$. 500. $\sqrt{5}-\sqrt{3}$.

501. $2\sqrt{5}-3\sqrt{2}$. 502. $4\sqrt{2}-3$.

503. $\frac{1}{2}\sqrt{5}-1$. 504. $2-\frac{1}{3}\sqrt{3}$.

505. $\frac{2}{3}\sqrt{2}-\frac{1}{4}$. 506. $2\sqrt{3}-\frac{3}{4}\sqrt{2}$.

507. $\sqrt{2}+1$. 508. $3-\sqrt{2}$.

509. $2\sqrt{2}-\sqrt{3}$. 510. $\sqrt{5}+\frac{1}{2}\sqrt{3}$.

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511. $x=\frac{1}{3}(1\pm\sqrt{7})$. 512. $x=6\pm2\sqrt{19}$.

513. $x=-\frac{3}{2}(3\pm\sqrt{3})$. 514. $x=\frac{1}{8}(27\pm\sqrt{57})$

515. $x = \frac{1}{4}$ or $-\frac{1}{2}$. 516. $x = 4$ or $\frac{1}{4}$.

517. $x = 1$ or $\frac{1}{81}$. 518. $x = -8$ or $-\frac{1}{8}$.

519. $x = \frac{1}{2}$ or $-\frac{1}{3}$. 520. $x = \pm 8$ or $\pm\frac{1}{8}$.

521. $x = 81$ or $\frac{1}{81}$. 522. $x = 4$.

523. $x = 3\frac{6}{7}$. 524. $x = \pm 5$ and $\pm 3\sqrt{2}$.

525. $x = \pm 4\sqrt{2}$ and ± 5. 526. $x = 10$ and $-11\frac{1}{15}$.

527. $x = 1$ and $\frac{1}{81}$. 528. $x = \frac{1}{8}(49 \pm \sqrt{97})$.

529. $x = 2\frac{2}{5}$ and $1\frac{4}{5}$. 530. $x = 64$ or 9.

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531. $33\sqrt{-1}$; $-\frac{1}{12}\sqrt{-3}$. 532. $4\sqrt{-1} - 6$; $x^3 + 1$.

533. $\sqrt{-1}$; $-\frac{3}{2}\sqrt{-1}$; $-\frac{8}{9}\sqrt{-2}$. 534. $x^6 + a^6$.

535. -12; $2x + 2\sqrt{x^2 + y^2}$

536. $6 + \sqrt{-5}$; $3 - 4\sqrt{-1}$; $1 + \sqrt{-1}$.

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537. $1 + 3x + 3x^2 + x^3$.

538. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

539. $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$.

540. $1 + 12x + 66x^2 + 220x^3 + 495x^4 + 792x^5 + 924x^6$
$+ 792x^7 + 495x^8 + 220x^9 + 66x^{10} + 12x^{11} + x^{12}$.

541. $1 + 8x + 24x^2 + 32x^3 + 16x^4$.

542. $1 + 20x + 160x^2 + 640x^3 + 1280x^4 + 1024x^5$.

543. $1 + 18x + 135x^2 + 540x^3 + 1215x^4 + 1458x^5 + 729x^6$.

544. $1 + 16x + 112x^2 + 448x^3 + 1120x^4 + 1792x^5 + 1792x^6$
$+ 1024x^7 + 256x^8$.

545. $1 + \frac{5x}{2} + \frac{5x^2}{2} + \frac{5x^3}{4} + \frac{5x^4}{16} + \frac{x^5}{32}$.

546. $1 + x + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{x^4}{256}$.

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547. $1 - x + x^2 - x^3 + x^4 -$ &c.

548. $1 + 3x + 9x^2 + 27x^3 + 81x^4 +$ &c.

549. $1 - 6x + 27x^2 - 108x^3 + 405x^4 -$ &c.

550. $1 + 6x + 24x^2 + 80x^3 + 240x^4 +$ &c.

551. $1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \frac{5}{16}x^4 +$ &c.

552. $1 - x + \frac{2}{3}x^2 - \frac{10}{27}x^3 + \frac{5}{27}x^4 -$ &c.

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553. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 +$ &c.

554. $1 - 2x - x^2 - \frac{4}{3}x^3 - \frac{7}{3}x^4 -$ &c.

555. $1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{35}{16}x^3 + \frac{315}{128}x^4 +$ &c.

556. $1 - \frac{7}{3}x^2 + \frac{14}{9}x^4 - \frac{14}{81}x^6 - \frac{7}{243}x^8 -$ &c.

557. $1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \frac{35}{243}x^4 +$ &c.

558. $1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^8 + \frac{35}{128}x^4 +$ &c.

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559. $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6.$

560. $a^9 - 9a^8x + 36a^7x^2 - 84a^6x^3 + 126a^5x^4 - 126a^4x^5$
$+ 84a^3x^6 - 36a^2x^7 + 9ax^8 - x^9.$

561. $32a^5 - 240a^4x + 720a^3x^2 - 1080a^2x^3 + 810ax^4 - 243x^5.$

562. $2187x^7 + 5103x^6y + 5103x^5y^2 + 2835x^4y^3 + 945x^3y$
$+ 189x^2y^5 + 21xy^6 + y^7.$

563. $\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \frac{5}{64}x^4 +$ &c.

564. $\frac{1}{27} + \frac{2}{27}x + \frac{8}{81}x^2 + \frac{80}{729}x^3 + \frac{80}{729}x^4 +$ &c.

565. $a^{-1} - a^{-2}bx + a^{-3}b^2x^2 - a^{-4}b^3x^3 + a^{-5}b^4x^4 -$ &c.

566. $a^{-2} + 2a^{-3}b^2x + 3a^{-4}b^4x^2 + 4a^{-5}b^6x^3 + 5a^{-6}b^8x^4 +$ &c.

567. $a^2 + 6a^{\frac{7}{3}}b^{\frac{1}{3}} + 21a^{\frac{8}{3}}b^{\frac{2}{3}} + 56a^3b + 126a^{\frac{10}{3}}b^{\frac{4}{3}} +$ &c.

568. $a - 3a^{\frac{4}{3}}b^{-\frac{1}{3}} + 6a^{\frac{5}{3}}b^{-\frac{2}{3}} - 10a^2b^{-1} + 15a^{\frac{7}{3}}b^{-\frac{4}{3}} -$ &c.

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b 3

569. $a^{\frac{1}{3}} - \frac{1}{3}a^{-\frac{2}{3}}x - \frac{1}{9}a^{-\frac{5}{3}}x^2 - \frac{5}{81}a^{-\frac{8}{3}}x^3 - \frac{10}{243}a^{-\frac{11}{3}}x^4 -$ &c.

570. $a^{\frac{2}{5}} - \frac{1}{5}a^{-\frac{8}{5}}x^2 - \frac{2}{25}a^{-\frac{18}{5}}x^4 - \frac{6}{125}a^{-\frac{28}{5}}x^6 - \frac{21}{625}a^{-\frac{38}{5}}x^8 -$ &c.

571. $a^{-1} + \frac{1}{5}a^{-6}x^5 + \frac{3}{25}a^{-11}x^{10} + \frac{11}{125}a^{-16}x^{15} + \frac{44}{625}a^{-21}x^{20} +$ &c.

572. $a^{\frac{4}{3}} - \frac{2}{3}a^{-\frac{2}{3}}x^2 - \frac{1}{9}a^{-\frac{8}{3}}x^4 - \frac{4}{81}a^{-\frac{14}{3}}x^6 - \frac{7}{243}a^{-\frac{20}{3}}x^8 -$ &c.

573. $a^{-\frac{3}{2}} + \frac{3}{4}a^{-\frac{7}{2}}x^2 + \frac{21}{32}a^{-\frac{11}{2}}x^4 + \frac{77}{128}a^{-\frac{15}{2}}x^6 + \frac{1155}{2048}a^{-\frac{19}{2}}x^8 +$ &c.

574. $a^{-\frac{1}{3}}x^{-\frac{1}{3}} + \frac{1}{3}a^{-\frac{4}{3}}x^{\frac{2}{3}} + \frac{2}{9}a^{-\frac{7}{3}}x^{\frac{5}{3}} + \frac{14}{81}a^{-\frac{10}{3}}x^{\frac{8}{3}} + \frac{35}{243}a^{-\frac{13}{3}}x^{\frac{11}{3}} +$ &c.

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575. $1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6$.

576. $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^6 + 10x^6 + 4x^7 + x^8$.

577. $1 + 5x + 5x^2 - 10x^3 - 15x^4 + 11x^5 + 15x^6 - 10x^7 - 5x^8 + 5x^9 - x^{10}$.

578. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

579. $1 + 8x + 36x^2 + 104x^3 + 214x^4 + 312x^5 + 324x^6 + 216x^7 + 81x^8$.

580. $a^3 - 3a^2b - 3a^2c + 3ab^2 + 6abc + 3ac^2 - b^3 - 3b^2c - 3bc^2 - c^3$.

581. $a^4 - 8a^3b + 4a^3c + 24a^2b^2 - 24a^2bc + 6a^2c^2 - 32ab^3 + 48ab^2c - 24abc^2 + 4ac^3 + 16b^4 - 32b^3c + 24b^2c^2 - 8bc^3 + c^4$.

582. $a^4 + 8a^3b - 4a^3c + 24a^2b^2 - 24a^2bc + 6a^2c^2 + 32ab^3 - 48ab^2c + 24abc^2 - 4ac^3 + 16b^4 - 32b^3c + 24b^2c^2 - 8bc^3 + c^4$.

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583. 17. 584. 75.

585. 793. 586. 9.

587. – 59. 588. – 31.

589. 14. 590. $-\frac{1}{2}$.

591. $-\frac{1}{3}$. 592. $-5\frac{5}{6}$.

593. $-13\frac{1}{2}$. 594. $-41\frac{1}{2}$.

595. 272. 596. 400.
597. 363. 598. 34750.
599. 16. 600. 0.
601. − 28. 602. − 275.
603. $16\frac{1}{4}$. 604. $-84\frac{1}{2}$.
605. $336\frac{2}{3}$. 606. − 84.

607. 12. 608. 5.
609. 20. 610. $-\frac{8}{77}$.
611. 5, 8, 11, 14. 612. − 2, − 6, − 10, − 14.
613. $3\frac{3}{5}$, $4\frac{1}{5}$, $4\frac{4}{5}$, $5\frac{2}{5}$, 6, $6\frac{3}{5}$, $7\frac{1}{5}$, $7\frac{4}{5}$, $8\frac{2}{5}$.
614. − 11, − 9, − 7, − 5, − 3, − 1, + 1.
615. 4, 15, 26, 37, 48, 59, 70, 81, 92, 103.
616. $-2\frac{3}{4}$, $-2\frac{1}{2}$, $-2\frac{1}{4}$, − 2, $-1\frac{3}{4}$, $-1\frac{1}{2}$, $-1\frac{1}{4}$, − 1.
617. − 2, $-1\frac{1}{4}$, $-\frac{1}{2}$, $\frac{1}{4}$, 1, $1\frac{3}{4}$, $2\frac{1}{2}$, $3\frac{1}{4}$, 4.
618. $-2\frac{4}{5}$, $-2\frac{1}{5}$, $-1\frac{3}{5}$, − 1, $-\frac{2}{5}$, $\frac{1}{5}$, $\frac{4}{5}$, $1\frac{2}{5}$, 2.
619. 5, 7, 9. 620. $-3\frac{1}{3}$, $3\frac{1}{3}$, 10.
621. $-\frac{3}{4}$, $\frac{1}{4}$, $1\frac{1}{4}$. 622. n^3.
623. £5. 3*s*.; £135. 4*s*. 624. 5 miles, 1300 yds.

625. 27. 626. 4096.
627. $38\frac{227}{512}$. 628. 4.
629. 324. 630. $-\frac{128}{243}$.
631. $\frac{2}{3}$. 632. $\frac{1}{2592}$.
633. $21\frac{7}{8}$. 634. $-\frac{8}{81}$.
635. $-2\frac{17}{32}$. 636. $15\frac{3}{16}$.

725. 15. 726. 3628800.
727. 6. 728. 4.

729. 126, 84. 730. 330, 330.
731. 3003. 732. 6.
733. 63. 734. 210, 84.
735. 50063860, 5006386. 736. 18.
737. 12. 738. 11.
739. 12. 740. 43092000.
741. 252, 1023. 742. 4, 56.
743. 12. 744. 3, 20.
745. 4 or 5, 126. 746. 10, 92378.

747. 5221, 203116.
748. 100101100, 102010, 10230, 2200, 1220.
749. 41104, 23420, 14641, 7571, 5954.
750. 1465. 752. 1295, 216.
751. 2304.
753. $1 + 14 + 244 + 4344 + 114144 + 2050544$,
$= 2214223$ in the senary scale,
$= 111111$ in the denary.
754. $111246 - 11334 + 1116 - 105$,
$= 100623$ in the septenary scale,
$= 17118$ in the denary.
755. 7. 756. 5.

757. $2^6 \times 100001 = 100001000000$ in the binary,
$4^3 \times 201 = 201000$ in the quaternary
and $64 \times 33 = 2112$ in the denary.

758. $1756 \times 345 = 701746$ in the octenary,
$1327 \times 274 = 381011$ in the nonary.
$1006 \times 229 = 230374$ in the denary.

759. 57264; *e*7*t*8.

760. 4112, 62*te*.

THE END.

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